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by

W. F. Chen*

ABSTRACT

Previous work on the solution of an eccentrically loaded beam-column under a concentrated load at the mid-span is extended. Expressions are first derived for the moment expressed explicitly in terms of curvature and thrust for various shapes of structural sections with or without the influence of residual stress. Using these results, interaction curves relating axial thrust, lateral load, and slenderness ratio with various values of end eccentricities are presented for the maximum load carrying capacity of the beam-columns. An approximate solution of the beam-column problem is then treated. Useful expressions for the interaction curves are obtained.

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1. INTRODUCTION

In an earlier paper [1], the author presented an analytical formulation for describing the elastic-plastic behavior of an eccentrically loaded beam-column under a concentrated load at the mid-span, Fig. 1. The formulation in Ref. [1] incorporates several features, the most significant of which are: (a) simple analytical expressions to fit the real moment-curvature-thrust curves for any complex shape of structural section to a high degree of approximation; (b) basic differential equations solved in terms of the curvature curve instead of the deflection curve.

In Ref. [1] the formulation is developed. Two special forms for the moment-curvature-thrust relationship are derived, one valid for a solid rectangular cross section (the expressions are exact for a perfectly plastic material), and the other valid for idealized wide-flange and box sections with very thin flange elements. For these two special cases, the elastic-plastic behavior of the beam-column problem, Fig. 1, is solved. In this paper, the application of the generalized moment-curvature-thrust relationship is extended to the following five cases:

(1) wide-flange section, strong axis bending
without the influence of residual stress,

(2) wide-flange section, strong axis bending
including the influence of residual stress,

(3) wide-flange section, weak axis bending
without the influence of residual stress,

(4) wide-flange section, weak axis bending including the influence of residual stress, and

(5) square tubular section without the influence of residual stress.

Using these results, the beam-column problem in Fig. 1 is solved. Numerical results for these five types of sections are then presented in the form of maximum load interaction curves. These curves relate axial thrust and lateral load with various values of slenderness ratio and eccentricity ratio of the beam-column. The influence of the shape of the cross section and the pattern of residual stress distribution on the maximum load carrying capacity of the beam-column should then become clearer. Finally, an approximate formulation of the beam-column problem is treated. The approximate solution includes the previously treated, eccentrically loaded column of solid rectangular cross section (Jezek's solution [2]) as a special case with even better accuracy so that it may be considered a generalization of Jezek's solution.

2. MOMENT-CURVATURE-THRUST RELATIONSHIPS

It is desirable to derive expressions for moment M in terms of axial force P and curvature Φ in a non-dimensional form so that they will be more convenient for numerical computations. For this purpose, the following initial yield quantities are defined:

$$M_y = \sigma_y S \ , \ P_y = \sigma_y A, \ , \ \Phi_y = \frac{2\epsilon_y}{h} \tag{1}$$

where σ_y is the yield stress in tension or compression and ϵ_y is the corresponding strain. Other quantities used are: area, A; elastic section modulus, S; and section height, h. One further defines the non-dimensional variables by

$$m = \frac{M}{M_y} \ , \ p = \frac{P}{P_y} \ , \ \varphi = \frac{\Phi}{\Phi_y} \tag{2}$$

A general m-φ-p curve of a common structural section with or without the influence of residual stress usually has the shape shown diagrammatically in Fig. 2. The curve may be divided into three regimes: elastic regime, primary plastic regime, and secondary plastic regime. They are separated by the points (m_1, φ_1) and (m_2, φ_2), as shown in the figure, with the moment asymptotically approaching the value m_{pc} as φ tends to infinity. The m-φ-p curve is assumed to be closely represented by the following expressions:

In the elastic regime,

$$m = a\,\varphi \tag{3}$$

valid for

$$0 \leq \varphi \leq \varphi_1 \tag{4}$$

In the primary plastic regime,

$$m = b - \frac{c}{\varphi^{1/2}} \tag{5}$$

valid for

$$\varphi_1 \leq \varphi \leq \varphi_2 \tag{6}$$

In the secondary plastic regime,

$$m = m_{pc} - \frac{f}{\varphi^2} \tag{7}$$

valid for

$$\varphi_2 \leq \varphi \tag{8}$$

where a, b, c, and f are arbitrary constants. These four constants can be evaluated easily by solving the two sets of two simultaneous equations which will arise if the particular values of φ_1 and φ_2 are inserted and the moments equated to the appropriate moments m_1 and m_2. These constants are

$$a = \frac{m_1}{\varphi_1} \tag{9}$$

$$b = \frac{m_2 - m_1\left(\frac{\varphi_1}{\varphi_2}\right)^{1/2}}{1 - \left(\frac{\varphi_1}{\varphi_2}\right)^{1/2}} \tag{10}$$

$$c = \frac{m_2 - m_1}{\left(\frac{1}{\varphi_1^{1/2}}\right) - \left(\frac{1}{\varphi_2^{1/2}}\right)} \tag{11}$$

$$f = \left(m_{pc} - m_2\right) \varphi_2^{\,2} \tag{12}$$

In the above equations, the points (m_1, φ_1) and (m_2, φ_2), and the value m_{pc}, may be treated as the arbitrary curve-fitting parameters. The parameters will be a function of axial force only, and will be referred to herein as the parameter functions. These parameter functions will be derived in what follows for the five types of cross section considered.

The choice of the curve-fitting parameter functions to fit an actual m-φ-p curve is not as difficult as it would seem to be. A general procedure to guide the choice of these parameter functions is suggested herein which involves the following steps:

(1) Since the parameter function is a function of the axial force p only, the simplest representation is clearly given by either a linear polynomial of p or a polynomial of second order of p.

(2) For a given set of m-φ-p curves which correspond to a particular shape of cross section as well as a particular pattern of residual stress distribution over a section, several sets of initial yield values of (m_1, φ_1) may be chosen, and the coefficients of the polynomials assumed in step 1 can be determined. In most cases, the linear polynomial assumption is found to be sufficient.

(3) The fully plastic moment m_{pc} of a section, which includes the reduction in moment-carrying capacity due to axial force, can be easily computed. The value of m_{pc} is, of course, independent of the residual stress distribution over a cross section.

(4) In many cases (particularly for sections without the influence of residual stress), the actual m-φ-p curve has an approximately constant slope 1

at small curvature and approaches the value m_{pc} at very large curvature (Fig. 2). The chosen procedures 1 to 3 then automatically guarantee that the actual m-φ-p curve for such a section will be closely represented by the approximate equations in these two regions.

(5) Several pairs of the values (m_a, φ_a) are chosen from the actual m-φ-p curve, and the corresponding coefficients of the polynomials, assumed in step 1 for (m_a, φ_a), can be determined.

(6) Compute the constants in Eqs. (9) to (12) and thus the approximate m-φ-p Eqs. (3) to (8).

(7) Compare the approximate m-φ-p curves with the actual m-φ-p curves. Repeat steps 5 and 6 if necessary.

To illustrate the derivation of the approximate m-φ-p expressions, consider the 8W31 cross section shown in Fig. 3. The actual m-φ-p curves are given by the dashed lines in the figure.

The expressions for the initial yield points m_1 and φ_1 are found, by inspection, to be:

$$m_1 = 1-p \tag{13}$$

$$\varphi_1 = 1-p \tag{14}$$

valid for all p.

For the 8W31 cross section, the approximate value of m_{pc} is found to be:

$$m_{pc} = \begin{cases} 1.11 - 2.64\ p^2 & 0 \leq p \leq 0.225 \\ 1.238 - 1.143\ p - 0.095\ p^2 & 0.225 \leq p \leq 1 \end{cases} \tag{15}$$

From Fig. 3 it can be seen that for the same increment of the constant axial force p there is almost a linear relationship between the bending moment m, curvature φ, and axial force p when the value of p is approximately greater than 0.2. Thus, the expressions for the points m_2 and φ_2 are assumed to be a linear polynomial of p when the value of p is greater than 0.225 and a polynomial of second order of p when the value of p is less than 0.225. The coefficients of these polynomials are then found by either inspection or by solving a set of simple simultaneous equations which will arise if the parameter functions for m_2 and φ_2 are forced to pass through certain specific points on the actual m-φ-p curves. These specific chosen points are shown by the small open-circles in Fig. 3. In this figure, two open circles are shown for a constant axial force, the first one for the point (m_1, φ_1) and the second one for the point (m_2, φ_2), respectively. The parameter functions for m_2 and φ_2 are found to be:

$$m_2 = 1 + 0.778\ p - 4.78\ p^2 \tag{16}$$

$$\varphi_2 = \frac{1}{1 - 3.70\ p + 8.40\ p^2} \qquad (17)$$

valid for

$$0 \leq p \leq 0.225$$

and

$$m_2 = 1.20\ (1-p) \qquad (18)$$

$$\varphi_2 = 2.20\ (1-p) \qquad (19)$$

valid for

$$0.225 \leq p \leq 1.0$$

Then, by substituting these parameter functions into Eqs. (9) to (12) yields

for

$$0 \leq p \leq 0.225 \qquad (20)$$

The constants which define the m-φ-p equation are:

$$a = 1 \qquad (21)$$

$$b = \frac{(1 + 0.778p - 4.78p^2) - (1-p)^{3/2}\ (1 - 3.70p + 8.40p^2)^{1/2}}{1 - (1-p)^{1/2}\ (1 - 3.70p + 8.40p^2)^{1/2}} \qquad (22)$$

$$c = \frac{p\ (1-p)^{1/2}\ (\ 1.778\ -\ 4.78p\ )}{1\ -\ (1-p)^{1/2}\ (\ 1\ -\ 3.70p\ +\ 8.40p^2\ )^{1/2}} \tag{23}$$

$$f = \frac{0.11\ -\ 0.778p\ +\ 2.14p^2}{(\ 1\ -\ 3.70p\ +\ 8.40p^2\ )^2} \tag{24}$$

for

$$0.225 \leq p \leq 1.0 \tag{25}$$

The constants which define the m-φ-p equation are:

$$a = 1 \tag{26}$$

$$b = 1.61\ (1-p) \tag{27}$$

$$c = 0.61\ (1-p)^{3/2} \tag{28}$$

$$f = 0.184\ (1-p)^2\ (1 + 1.5p - 2.5p^2) \tag{29}$$

Equations (3) to (8) will then give the approximate expressions for the 8W31 m-φ-p curves. The analytically obtained expressions are compared in Fig. 3 with the actual m-φ-p curves; a good agreement is observed.

Table 1 lists all parameter functions of the approximate m-φ-p expressions for the five cross sections considered. For ease of comparison, the parameter functions for the special case of a solid rectangular section is also listed in the Table. The approximate expressions and the actual m-φ-p curves for a wide-flange shaped cross section which includes the influence of the residual stress pattern indicated are compared in Fig. 4. Additional comparisons for other shapes of sections are given in Appendix I. The comparison shows that the approximate m-φ-p equations are sufficiently accurate for practical use.

3. NUMERICAL SOLUTIONS OF A BEAM-COLUMN PROBLEM

The beam-column problem under consideration is shown in Fig. 1. It is assumed that the failure of the beam-column is due to excessive bending in the plane of the applied loads. It is further assumed that the axial force P at the ends of the beam-column is applied first and maintained at a constant value as the lateral load Q increases or decreases. The moment-curvature-thrust relationship is assumed to be described by Eqs. (3) to (8). The analytical solution of this problem has been given in [1]. Details of the method of computation have been programmed for a digital computer [3]. Thus, the numerical results for the beam-column problem can be obtained directly.

Numerical results of the beam-column are obtained for the five types of cross sections considered in Section 2. The results are presented in the form of maximum load carrying capacity interaction curves in Figs. 5 to 14. For each type of cross section, two sets of interaction curves are shown, one for a beam-column with zero end eccentricity ($e/r = 0$), and the other for a beam-column with $e/r = 0.5$. In each figure, slenderness ratios ranging from 20 to 160 were considered in intervals of 20. Each interaction curve was plotted in an increment of 0.05p. In general, the results are calculated for structural steel with E = 30,000 ksi and $\sigma_y = 34$ ksi. Since the interaction curves are non-dimensionalized, they can be directly used in analysis and design computations*.

The interaction curves determined by the analytical solution for the beam-column with $e/r = 0$ are compared in Fig. 15 with those obtained previously by the numerical integration procedure [4]. Good agreement is observed. To see the effect of cross section shape on the maximum strength of a beam-column of the type considered herein, Fig. 16 gives six comparable interaction curves for two values of $\ell/r = 40$ and 100. The wide-flange cross section, subjected to weak axis bending without the influence of residual stress, gives an upper limit on the two sets of interaction curves. The wide-flange cross section, subjected to strong axis bending including the influence of residual stress, gives a lower limit except in the low lateral force range. The difference between upper and lower limit is seen to be quite significant when the values of slenderness ratio of the beam-column are small, but the difference becomes relatively small when the values of slenderness ratio are increased. It may be concluded from this comparison that the interaction curves derived for the particular shape, a wide-flange cross section subjected to strong axis bending including the influence of residual stress, in general, give a conservative result among all types of cross section considered.

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*The interaction curves can also be applied to steels of other yield stress levels by simply substituting an equivalent slenderness ratio [1]

$$\left(\frac{\ell}{r}\right)_{equ} = \left(\frac{\ell}{r}\right)\sqrt{\frac{34}{\sigma_y}}$$

In Fig. 17 are shown two sets of column curves for a similar kind calculated for $e/r = 0$ and 0.5, respectively. It can be seen that the column curves are relatively insensitive to the shapes of column section. However, the column strength for an axially loaded column ($e/r = 0$) is considerably reduced when the influence of residual stress is included.

4. APPROXIMATE SOLUTION OF THE ECCENTRICALLY LOADED COLUMN PROBLEM

In Ref. [2] approximate interaction equations of an eccentrically loaded column of rectangular cross section (Fig. 1, set $Q = 0$) are presented. The solution is based upon three simplifications: (a) the axis of the deflected column is assumed to be the half wave of a sine curve; (b) the equilibrium condition is established only at mid-height of the column; (c) the stress-strain relation is assumed to be elastic-perfectly plastic so that a simple moment-curvature-thrust relationship for a rectangular cross section can be obtained. It was shown in Ref. [2] that, in general, the values of the maximum strength computed from the approximate equations are higher than the values determined from the exact solution using the real stress-strain curve, but the difference is usually small. For all practical purposes, these approximate equations give satisfactory results.

The same idea will be extended in what follows for columns with more complicated shapes of cross section. Approximate interaction equations for the eccentrically loaded column problem of Fig. 1 ($Q = 0$) are given in the following paragraphs, and the solution for the general case of Fig. 1 is presented in Section 5.

Instead of the deflection curve being used for calculating the maximum strength of the stability problem, the problem can be solved differently by using the curvature curve, as was done in the two proceeding papers [1,5]. It is assumed that the curvature curve of the column takes the form

$$\Phi = \Phi_o + (\Phi_m - \Phi_o) \sin \frac{\pi x}{\ell} \tag{30}$$

where Φ_o and Φ_m denote the curvature at the ends and at the mid-height of the column, respectively. By integrating the curvature curve twice and fulfilling the condition that the deflections at the ends of the column are zero, one concludes that

$$y = \frac{\Phi_o}{2} x \; (\ell - x) + (\Phi_m - \Phi_o) \frac{\ell^2}{\pi^2} \sin \frac{\pi x}{\ell} \tag{31}$$

The curve is symmetric with respect to the mid-height deflection y_m. It satisfies not only the deflection conditions but also the curvature conditions at the ends of the column. By assuming the end moment $M_o = EI \; \Phi_o$ where EI represents the flexural rigidity of the column in the plane of bending, the deflection at mid-height is

$$y_m = \frac{\ell^2}{\pi^2} \left[\Phi_m + \frac{M_o}{EI} \left(\frac{\pi^2}{8} - 1 \right) \right] \tag{32}$$

Equilibrium at the mid-height of the column requires that $M_o = M_m - P \; y_m$ where M_m denotes the internal moment at the mid-height section. Using Eq. (32) and introducing the following notation

$$p_E = \frac{P_E}{P_y} = \frac{\pi^2 \; E}{\sigma_y \; (\ell/r)^2} \tag{33}$$

The non-dimensional resulting equilibrium equation is then

$$m_o = \frac{m_m - \frac{p}{p_E}\varphi_m}{\left[1 + \left(\frac{\pi^2}{8} - 1\right)\frac{p}{p_E}\right]} \tag{34}$$

Equation (34) is the analytical expression for the (m_o, φ_m)-curves which can be drawn for various values of the parameter p/p_E. These curves are the same nature as the (m_o, y_m)-curves used previously in much of the pertinent literature. The apex of each curve, defined by $dm_o/d\varphi_m = 0$, determines the maximum strength of the column, which is:

$$\frac{dm_m}{d\varphi_m} = \frac{p}{p_E} \tag{35}$$

Since the pertinent m-φ-p relations are given by Eqs. (3) to (8), the critical value of φ_m, corresponding to the critical value of m_o, may be easily derived*. It should be noted, however, that at the point $\varphi = \varphi_1$ or $\varphi = \varphi_2$ (Fig. 2) infinitely many slopes may be associated with a given state of (m_1, φ_1) or (m_2, φ_2) and the corresponding critical values of φ_m and m_o are φ_1 or φ_2 and m_1 or m_2, respectively.

Introducting Eq. (35) into Eq. (34) and using the basic m-φ-p relations [Eqs. (3) to (8)], the resulting equilibrium equations are then

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*The derivatives (Eq. 35), which will be here obtained from analytic expressions, can be also obtained graphically from actual m-φ-p curves. Thus, a better result may be expected.

In the regime (m_1, φ_1)

$$m_o = \frac{m_1 - \left(\frac{p}{p_E}\right)\varphi_1}{\left[1 + \left(\frac{\pi^2}{8} - 1\right)\frac{p}{p_E}\right]} \tag{36}$$

valid for

$$\frac{c}{2\varphi_1^{3/2}} \leq \frac{p}{p_E} \leq 1 \tag{37}$$

In the primary plastic regime

$$m_o = \frac{b - 3\,c^{3/2}\left[\frac{1}{4}\frac{p}{p_E}\right]^{1/3}}{\left[1 + \left(\frac{\pi^2}{8} - 1\right)\frac{p}{p_E}\right]} \tag{38}$$

valid for

$$\frac{c}{2\varphi_2^{3/2}} \leq \frac{p}{p_E} \leq \frac{c}{2\varphi_1^{3/2}} \tag{39}$$

In the regime (m_2, φ_2)

$$m_o = \frac{m_2 - \left(\frac{p}{p_E}\right)\varphi_2}{\left[1 + \left(\frac{\pi^2}{8} - 1\right)\frac{p}{p_E}\right]} \tag{40}$$

valid for

$$\frac{2f}{\varphi_2^{3}} \leq \frac{p}{p_E} \leq \frac{c}{2\varphi_2^{3/2}} \tag{41}$$

In the secondary plastic regime

$$m_o = \frac{m_{pc} - 3f^{1/3}\left[\frac{1}{2}\frac{p}{p_E}\right]^{2/3}}{\left[1 + \left(\frac{\pi^2}{8} - 1\right)\frac{p}{p_E}\right]} \tag{42}$$

valid for

$$0 \leq \frac{p}{p_E} \leq \frac{2f}{\varphi_2^{\,3}} \tag{43}$$

These four equations [Eqs. (36), (38), (40), and (42)] represent the relation between the critical end moment m_o and the axial force p for any given value of the parameter p_E from Eq. (33).

For the particular case of a rectangular cross section, the slopes of the generalized m-φ-p curve at the points $\varphi = \varphi_1$ and $\varphi = \varphi_2$ are continuous (Fig. 2); thus, the values of the upper and lower limits of Eq . (37) and (41) are identical, and the other two interaction equations reduce to

In the primary plastic regime

$$m_o = \frac{3\,(1 - p)\left[1 - \left(\frac{p}{p_E}\right)^{1/3}\right]}{\left[1 + \left(\frac{\pi^2}{8} - 1\right)\frac{p}{p_E}\right]} \tag{44}$$

valid for

$$(1 - p)^3 \leq \frac{p}{p_E} \leq 1 \tag{45}$$

In the secondary plastic regime

$$m_o = \frac{3\left[1 - p^2 - \left(\frac{p}{p_E}\right)^{2/3}\right]}{2\left[1 + \left(\frac{\pi^2}{8} - 1\right)\frac{p}{p_E}\right]} \tag{46}$$

valid for

$$0 \leq \frac{p}{p_E} \leq (1 - p)^3 \tag{47}$$

Now Jezek's interaction equations [2], which only apply to eccentrically loaded columns of rectangular cross section, correspond to the particular case of the present formulation when there is no end curvature, $\Phi_o = 0$ in Eq. (30). It can then be shown that the factor $\left[1 + (\pi^2/8 - 1)\, p/p_E\right]$ which appeared in the denominator of Eqs. (44) and (46) will disappear and Eqs. (44) to (46) become Jezek's familiar formulas*. The value of the factor $\left[1 + (\pi^2/8 - 1)\, p/p_E\right]$ must lie between the limits 1 and 1.23 $(0 \leq p/p_E \leq 1)$. It follows that Eqs. (44) and (46) will always give values of the critical moment which are smaller than the values of the critical moment computed from Jezek's formulas unless the axial

*The present form of the Jezek's formulas was first brought to the attention of the author in the book by T. V. Galambos, "Structural Members and Frames", Prentice-Hall, Inc., New Jersey, pp. 260-264, 1968.

force happens to be zero. In the following discussion, it will prove convenient to denote the factor

$$D = \left[1 + \left(\frac{\pi^2}{8} - 1 \right) \frac{p}{p_E} \right] \tag{48}$$

which appears in the denominator of all the above interaction equations. These interaction equations will now be divided into two kinds, one of which is termed "present approximate solution" and the other is termed "generalized Jezek's solution", depending upon whether D is assigned the value of Eq. (48) or the value of unity, respectively.

At this point, it is necessary to discuss the errors involved in the analytical results of the approximate interaction equations derived previously. In Figs. 18 and 19, two series of such interaction curves are compared. These were calculated for a solid rectangular cross section and for a wide-flange shaped cross section. (Further, curves for other shapes of cross section are presented in Appendix II). These curves are:

Curve 1. "Exact Solution" -- The values of the critical moment are determined from the real stress-strain curve using the method of numerical integration (see Ref. 6, for example).

Curve 2. "Present Exact Solution" -- Analytical solutions are developed in Ref. 1. The m-φ-p curves are assumed to be closely represented by Eqs. (3) to (8). The actual m-φ-p curves presented in Section 2 are computed from an idealized stress-strain diagram (elastic-perfectly plastic), and the corresponding parameter functions are given in Table 1.

Curve 3. "Present Approximate Solution" -- The values of the critical moment are computed from Eqs. (36) to (43) using Table 1.

Curve 4. "Generalized Jezek's Solution" -- The values of the critical moment are computed from Eqs. (36) to (43) using Table 1 but assigning the expression D (Eq. (48)) the values of unity.

Although no exact curves (Curve 1) are yet available to compare for other types of shapes except the rectangular section*, one can still make some comparisons and observe some trends. All curves in Figs. 18 and 19 are seen to fall into a relatively narrow band. The curves for the present exact solution (Curve 2) are seen to be closely represented by the generalized Jezek's solution (Curve 4); while the exact curves (rectangular section only) are seen to be better represented by the present approximate solution (Curve 3). It may be concluded from this comparison that the approximate interaction equations derived herein [Eqs. (36) to (43)] may be considered a suitable basis for a method of design for eccentrically loaded columns.

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*Most of the numerical solutions available were based on the m-φ-p curves which were computed from the idealized stress-strain diagram. Thus, these results are found to be very close to "Curve 2".

It is of interest to note that the expression D, as given by Eq. (48), may be interpreted as a correction factor for the purpose of improving the accuracy of Jezek's well-known solution. A different way to improve Jezek's solution has been discussed thoroughly by Bleich [6].

5. APPROXIMATE SOLUTION OF THE BEAM-COLUMN PROBLEM

The extension of the approximate theory discussed in the previous section to the more general case of the beam-column shown in Fig. 1 is evident. The procedure will be the same as before. The condition of equilibrium is

$$Q = \frac{4}{\ell} (M_m - M_o) - \frac{4 P \ell}{\pi^2} \left[\Phi_m + \left(\frac{\pi^2}{8} - 1 \right) \Phi_o \right] \tag{49}$$

representing the (Q, Φ_m)-curves. The condition $dQ/d\Phi_m = 0$ then gives the value of Φ_m which is associated with the critical lateral load Q. It is found that the non-dimensional form of the critical condition is identical with that of Eq. (35) obtained in the previous section. Introducing the m-φ-p expressions [Eqs. (3) to (8)] into Eq. (35) and substituting the value of Φ_m thus obtained into Eq. (49) yield the critical lateral load of the beam-column. Denoting the shape factor of a section about the axis of bending by α and introducing the quantity $Q_p = 4\alpha M_y/\ell$ which is the plastic limit load of the beam-column according to simple plastic theory, the non-dimensional form of the critical lateral load $q = Q/Q_p$ is derived:

In the regime (m_1, φ_1)

$$\alpha q = m_1 - \left(\frac{p}{p_E} \right) \varphi_1 - m_o D \tag{50}$$

valid for

$$\frac{c}{2\varphi_1^{3/2}} \leq \frac{p}{p_E} \leq 1 \tag{51}$$

<u>In the primary plastic regime</u>

$$\alpha q = b - 3\, c^{2/3} \left[\frac{1}{4}\,\frac{p}{p_E}\right]^{1/3} - m_o\, D \tag{52}$$

valid for

$$\frac{c}{2\varphi_2^{3/2}} \leq \frac{p}{p_E} \leq \frac{c}{2\varphi_1^{3/2}} \tag{53}$$

<u>In the regime $(m_2,\ \varphi_2)$</u>

$$\alpha q = m_2 - \left(\frac{p}{p_E}\right) \varphi_2 - m_o\, D \tag{54}$$

valid for

$$\frac{2f}{\varphi_2^{3}} \leq \frac{p}{p_E} \leq \frac{c}{2\varphi_2^{3/2}} \tag{55}$$

<u>In the secondary plastic regime</u>

$$\alpha q = m_{pc} - 3\, f^{1/3} \left[\frac{1}{2}\,\frac{p}{p_E}\right]^{2/3} - m_o\, D \tag{56}$$

valid for

$$0 \leq \frac{p}{p_E} \leq \frac{2f}{\varphi_2^{\ 3}} \tag{57}$$

in which D is a function of p/p_E, as defined in Eq. (48).

Equations (36) to (43) may be interpreted as the special case of Eqs. (50) to (57) when q = 0.

The accuracy of the approximate interaction equations is checked in Figs. 20 and 21 against the interaction curves determined by the present exact solution, samples of which are already shown in Figs. 5-14. The curves are calculated for a wide-flange cross section including the influence of residual stress with the eccentricity ratios shown in the figures. (Other shapes are compared in Appendix III). From these curves, it is observed that for a constant slenderness ratio the deviation between the values of the critical lateral load q computed from the approximate theory and the values determined in Section 3 of the present paper becomes appreciable for values of p in the lower value range but decreases rapidly when the values of q are decreased.

Although the interaction equations discussed previously do not precisely give the maximum strength of the beam-column, they represent a reasonable estimate of the strength for the types of shapes considered. Even though the present exact curves shown in Fig. 21 are seen to be better represented by the generalized Jezek's solution (assigning the value of $D = 1$), it is believed, however, that the present form of Eqs. (50) to (57) will agree better with the exact curves using the real stress-strain diagram, as was already found to be the case for eccentrically loaded columns. (Note: in Fig. 20, $e/r = 0$, thus, curves marked 3 and 4 are identical).

6. CONCLUSIONS

The solutions presented here are a continuation of an investigation reported in Ref. [1]. The beam-column problem solved here is not only of great practical interest but, because of the present success, the general theory presented in [1] is also of great significance for further practical applications. Furthermore, the numerous diagrams presented brought further insight into the behavior of laterally loaded columns as influenced by shape of the column section, slenderness ratio, eccentricity, and residual stress distribution.

The analytical results of the approximate theory discussed, which is essentially an extension of Jezek's concept, should prove both interesting and useful. They are derived in a rather simple manner, and the resulting interaction curves are found to agree rather well with the exact results. Therefore, it appears that these approximate interaction equations may be considered as suitable bases for a method of design for eccentrically as well as laterally loaded compression members.

7. ACKNOWLEDGEMENTS

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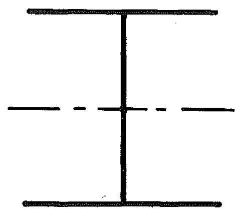
TABLE 1 -- PARAMETER FUNCTIONS FOR THE APPROXIMATE m-φ-p EXPRESSIONS

(1)



Strong Axis Bending No Residual Stress

For all p

$$m_1 = 1 - p$$
$$\varphi_1 = 1 - p$$

For

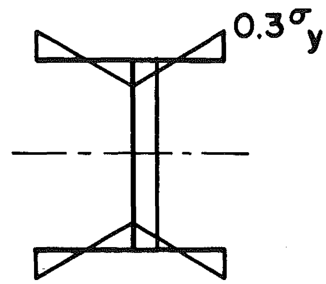
$$0 \le p \le 0.225$$

$$m_{pc} = 1.11 - 2.64p^2$$
$$m_2 = 1 + 0.778p - 4.78p^2$$
$$\varphi_2 = \frac{1}{1 - 3.7p + 8.4p^2}$$

$$0.225 \le p \le 1$$

$$m_{pc} = 1.238 - 1.143p - 0.095p^2$$
$$m_2 = 1.20\ (1-p)$$
$$\varphi_2 = 2.20\ (1-p)$$

(2)



Strong Axis Bending with Residual Stress

For

$$0 \le p \le 0.8$$

$$m_1 = 0.9 - p$$
$$\varphi_1 = 0.9 - p$$

$$0.8 \le p \le 1$$

$$m_1 = -1.1 + 3.1p - 2p^2$$
$$\varphi_1 = 3.3 - 8p + 5p^2$$

For

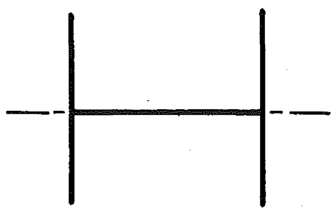
$$0 \le p \le 0.225$$

$$m_{pc} = 1.11 - 2.64p^2$$
$$m_2 = 0.9 + 1.94p - 9.4p^2$$
$$\varphi_2 = \frac{1}{1.11 - 7.35p + 29.2p^2}$$

$$0.225 \le p \le 1$$

$$m_{pc} = 1.238 - 1.143p - 0.095p^2$$
$$m_2 = 1.1\ (1-p)$$
$$\varphi_2 = 1.3 - p$$

(3)



Weak Axis Bending
No Residual Stress

For all p

$$m_1 = 1 - p$$
$$\varphi_1 = 1 - p$$

For

$$0 \leq p \leq 0.4$$

$$m_2 = 1 + 1.5p - 2.5p^2$$
$$\varphi_2 = \frac{1}{1 - 1.57p + 0.725p^2}$$

$$0.4 \leq p \leq 1$$

$$m_2 = 0.85 + 2.03p - 2.88p^2$$
$$\varphi_2 = \frac{1}{0.368 + 0.645p - 0.862p^2}$$

For

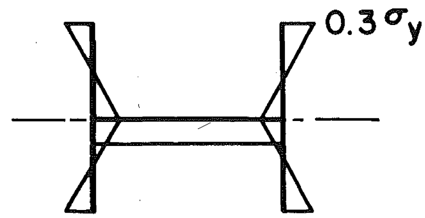
$$0 \leq p \leq 0.252$$

$$m_{pc} = 1.51\ (1 - 0.185p^2)$$

$$0.252 \leq p \leq 1$$

$$m_{pc} = 2.58\ (0.52 + p)\ (1 - p)$$

(4)



Weak Axis Bending
with Residual Stress

For

$$0 \leq p \leq 0.4$$

$$m_1 = 0.9 - p$$
$$\varphi_1 = 0.9 - p$$
$$m_2 = 0.9 + p - 2.5p^2$$
$$\varphi_2 = \frac{1}{1.11 - 2.11p + 2.81p^2}$$

$$0.4 \leq p \leq 1$$

$$m_1 = 0.567 + 0.1p - 0.667p^2$$
$$\varphi_1 = 0.5$$
$$m_2 = 1 + 0.25p - 1.25p^2$$
$$\varphi_2 = \frac{1}{1.3 - 2.45p + 2.45p^2}$$

For

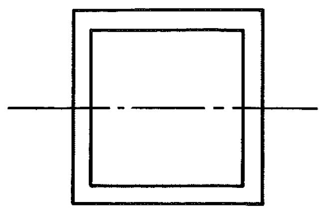
$$0 \leq p \leq 0.252$$

$$m_{pc} = 1.51 - 0.28p^2$$

$$0.252 \leq p \leq 1$$

$$m_{pc} = 2.58\ (0.52 + p)\ (1 - p)$$

(5)



Square Tubular Section
No Residual Stress

For all p

$$m_1 = 1 - p$$
$$\varphi_1 = 1 - p$$

For

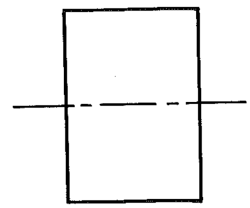
$$0 \leq p \leq 0.467$$

$$m_{pc} = 1.20 - 1.60p^2$$
$$m_2 = 1 + 0.9p - 3.25p^2$$
$$\varphi_2 = \frac{1}{1 - 2.5p + 4.17p^2}$$

$$0.467 \leq p \leq 1$$

$$m_{pc} = 1.51 - 1.31p - 0.2p^2$$
$$m_2 = 1.40\ (1 - p)$$
$$\varphi_2 = 2.50\ (1 - p)$$

(6)



Solid Rectangular
Section

For all p

$$m_1 = 1 - p$$
$$\varphi_1 = 1 - p$$

$$m_2 = \frac{1}{1 - p}$$
$$\varphi_2 = \frac{1}{1 - p}$$

$$m_{pc} = \frac{3}{2}\ (1 - p^2)$$

9. FIGURES

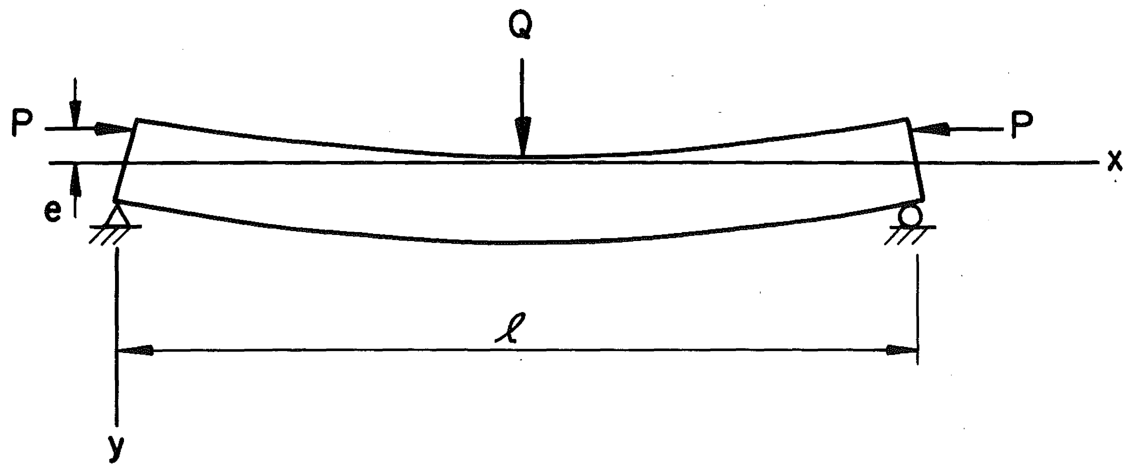


Fig. 1 An Eccentrically Loaded Beam-Column

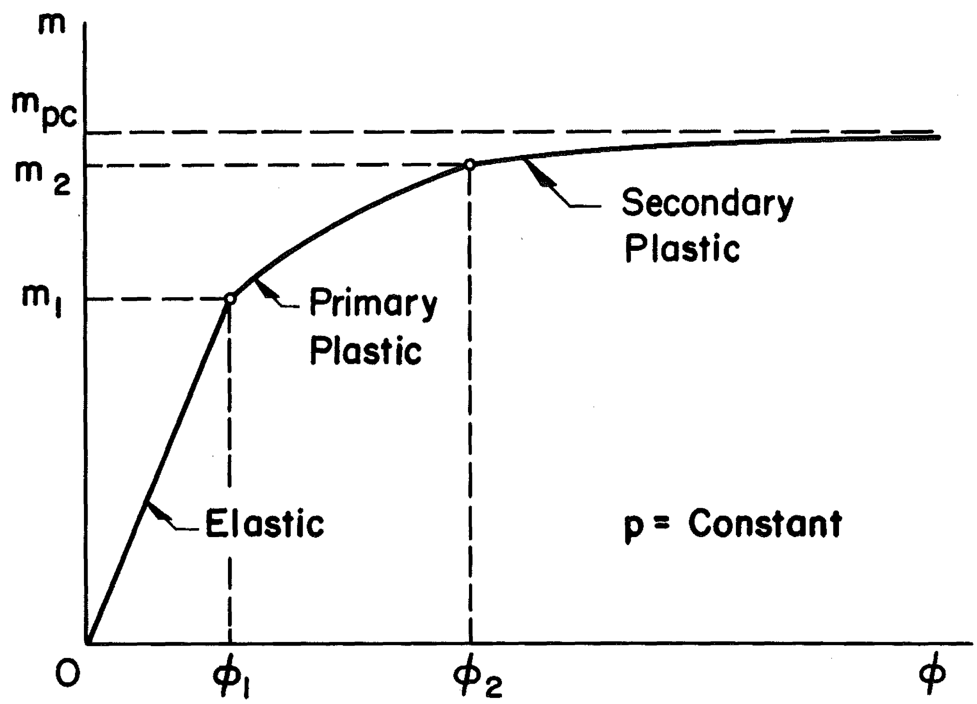


Fig. 2 Moment-Curvature-Thrust Relationship
for a Common Structural Section

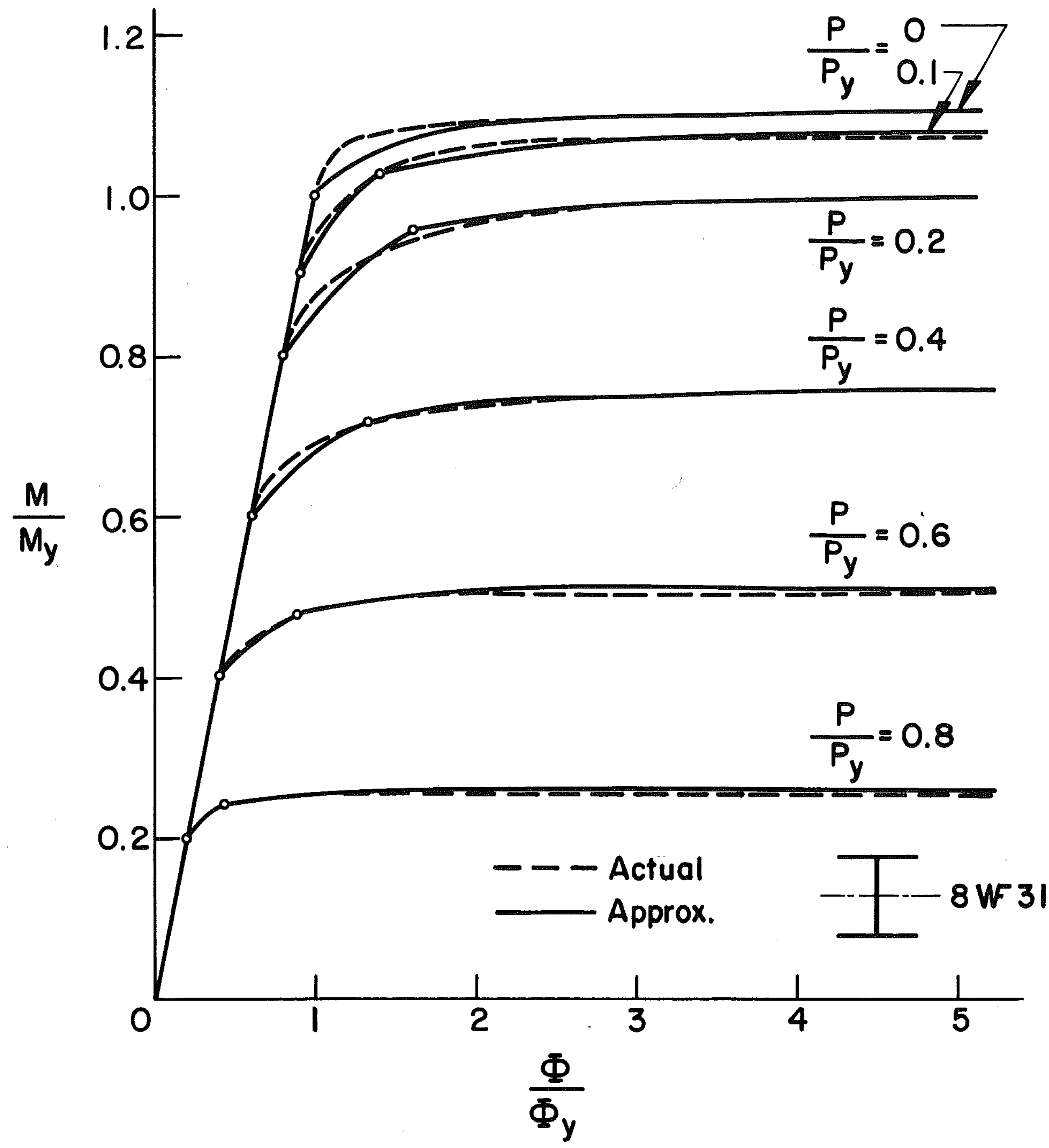


Fig. 3 Moment-Curvature-Thrust Relationship, Neglecting Influence of Residual Stress (Actual Curves Shown Dashed)

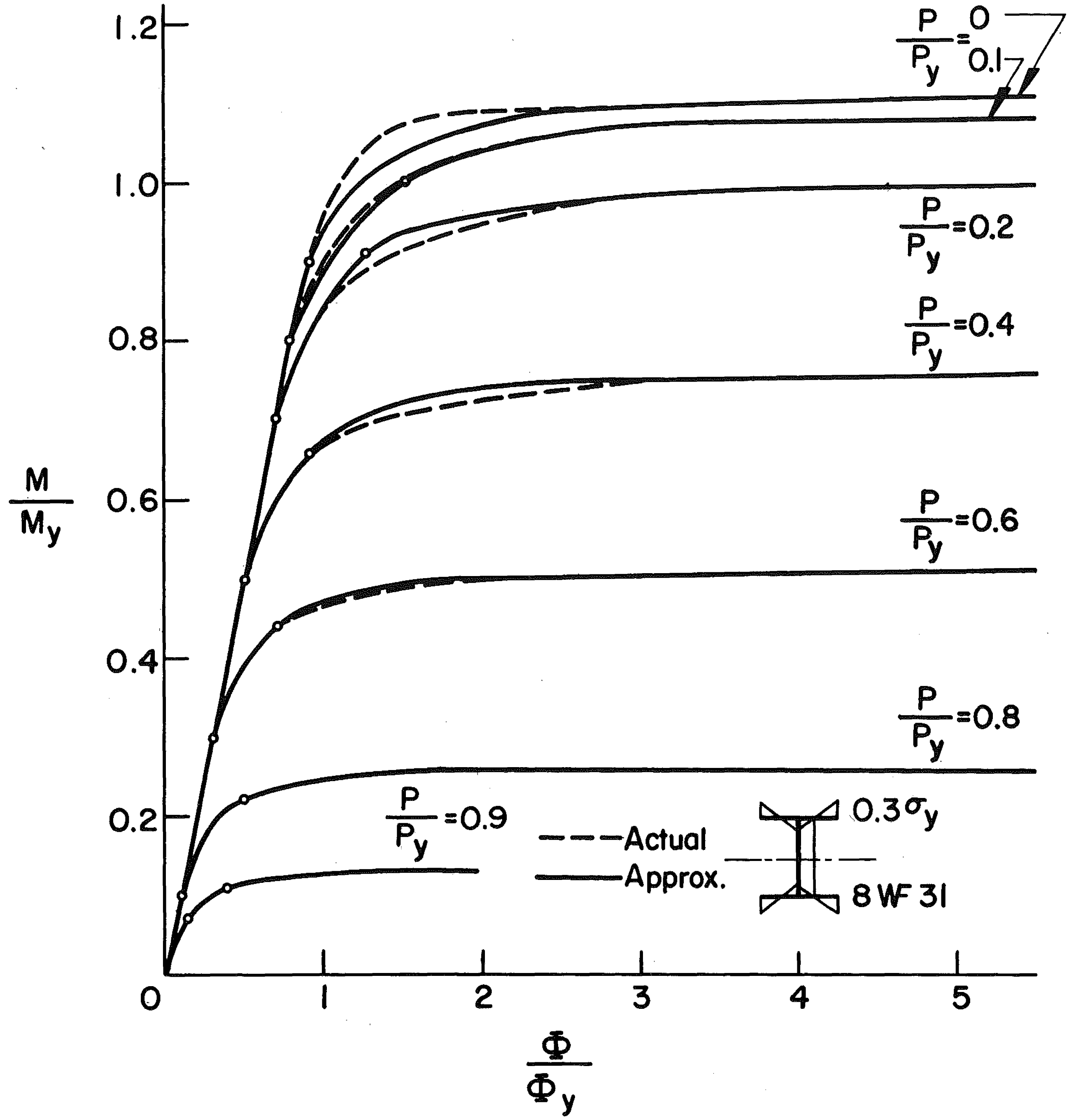


Fig. 4 Moment-Curvature-Thrust Relationship, Including Influence of Residual Stress (Actual Curves Shown Dahsed)

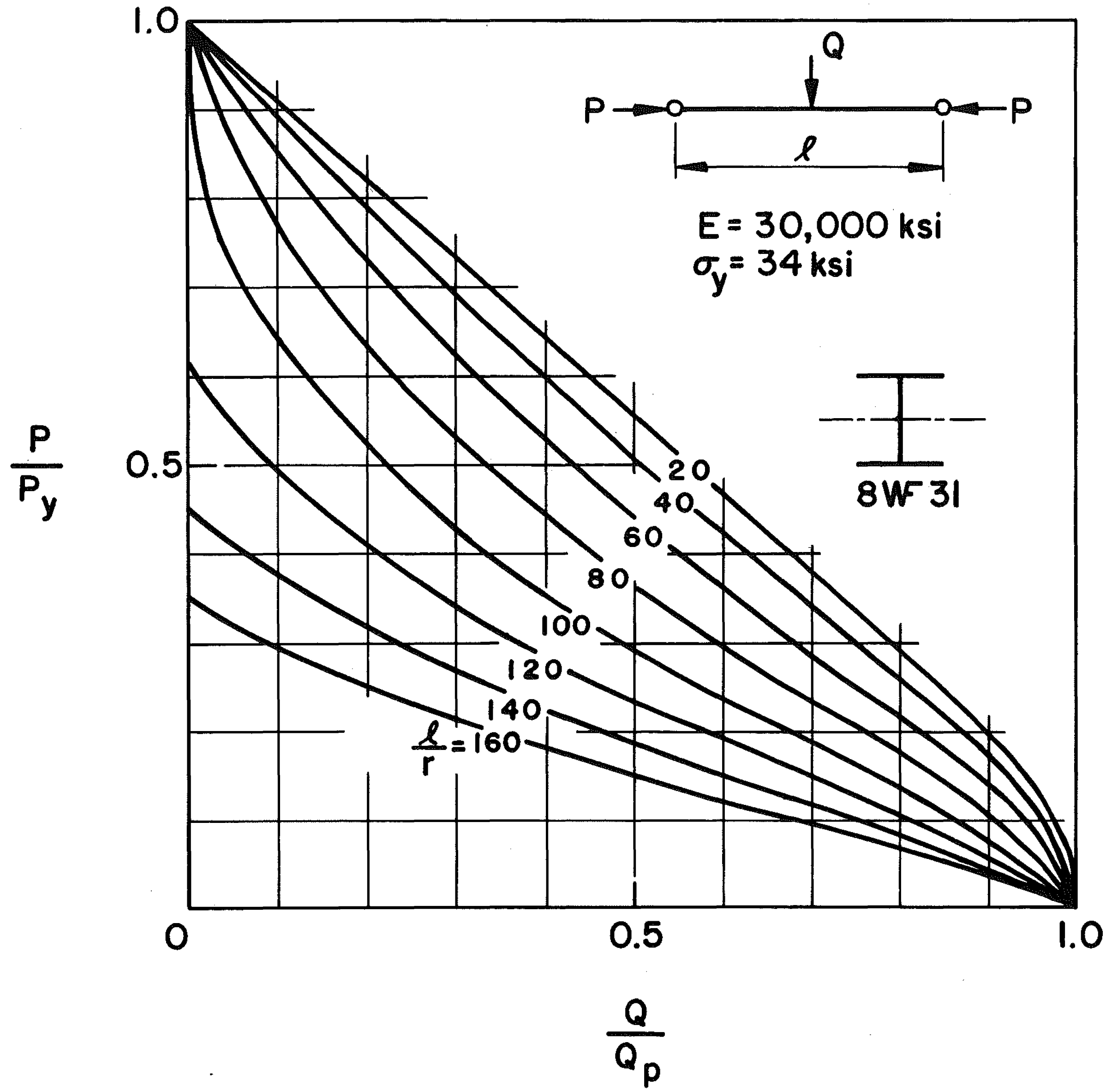


Fig. 5 Interaction Curves for a Wide-Flange Section, Neglecting Influence of Residual Stress (e/r = 0)

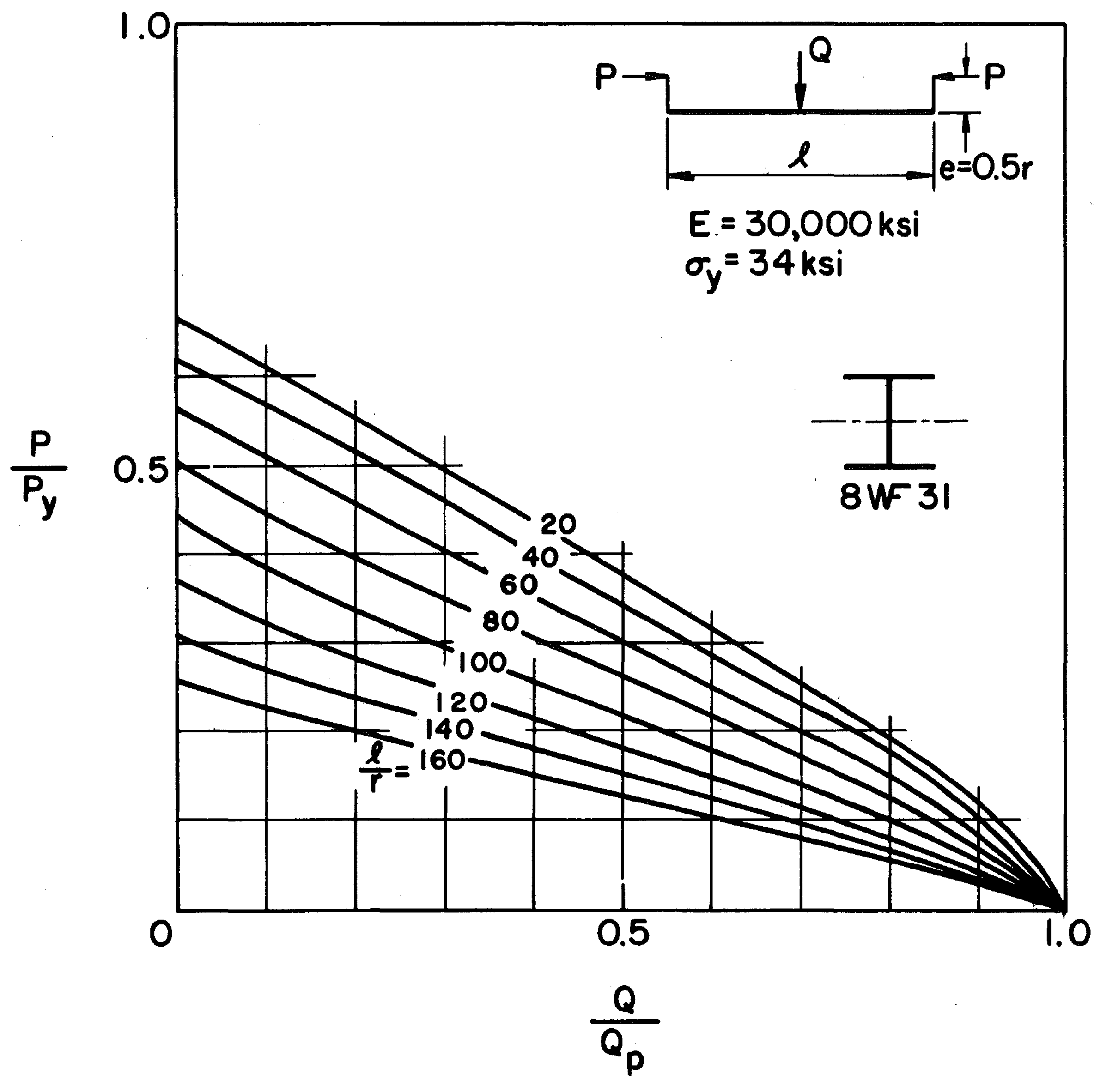


Fig. 6 Interaction Curves for a Wide-Flange Section, Neglecting Influence of Residual Stress (e/r = 0.5)

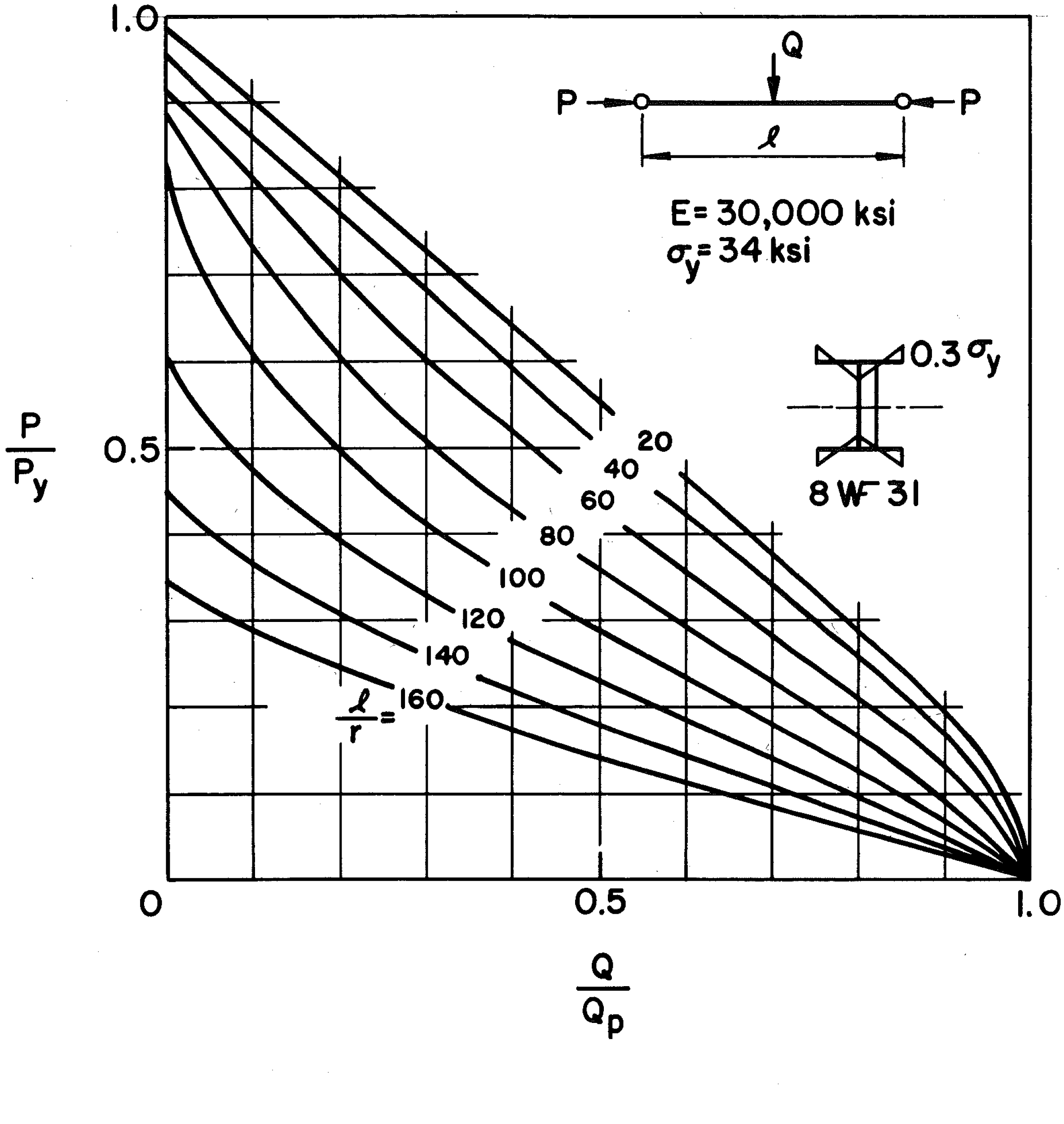


Fig. 7 Interaction Curves for a Wide-Flange Section, Including Influence of Residual Stress (e/r = 0)

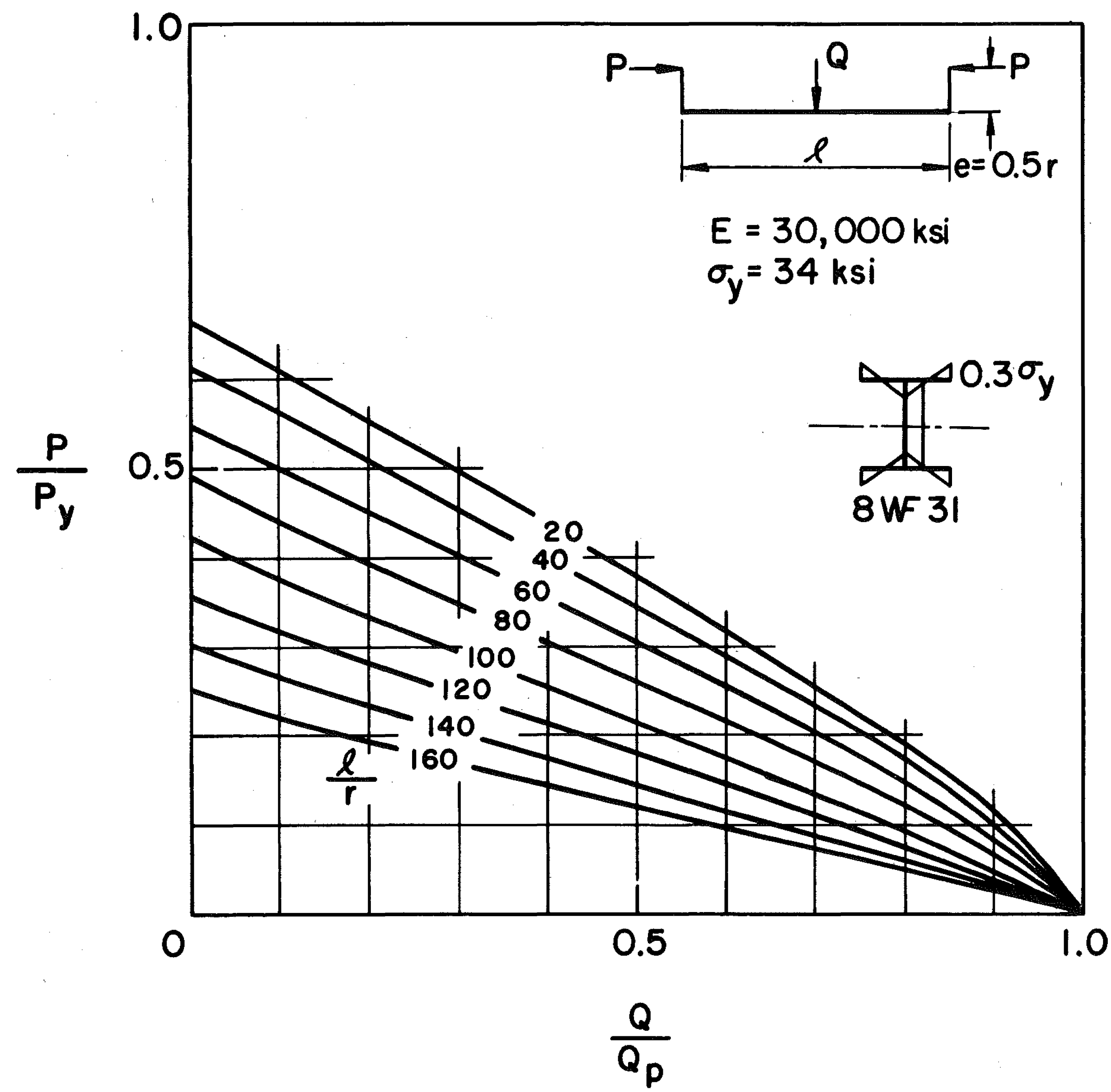


Fig. 8 Interaction Curves for a Wide-Flange Section, Including Influence of Residual Stress ($e/r = 0.5$)

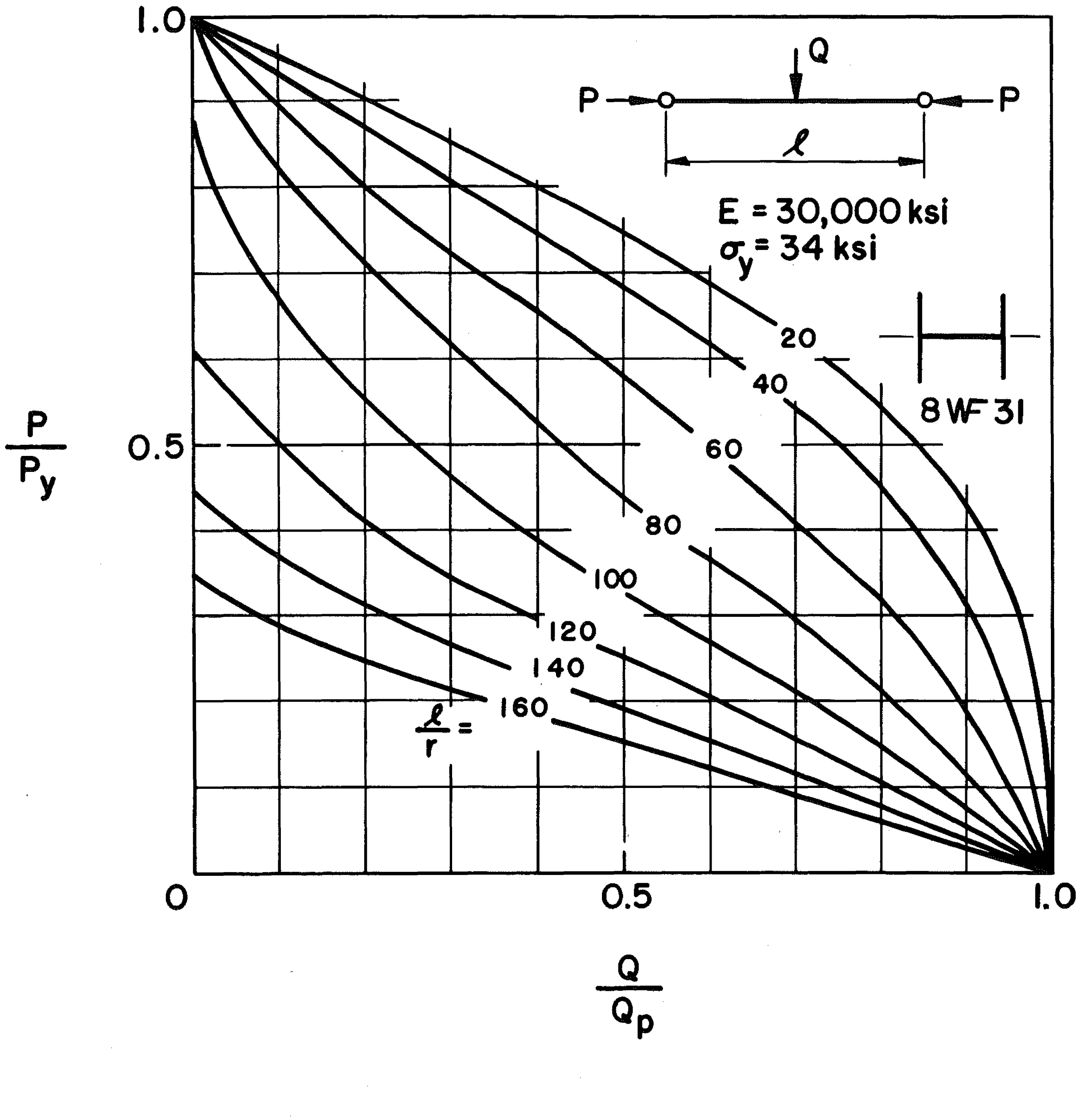


Fig. 9 Interaction Curves for a Wide-Flange Section,
Weak Axis Bending,
Neglecting Influence of Residual Stress
(e/r = 0)

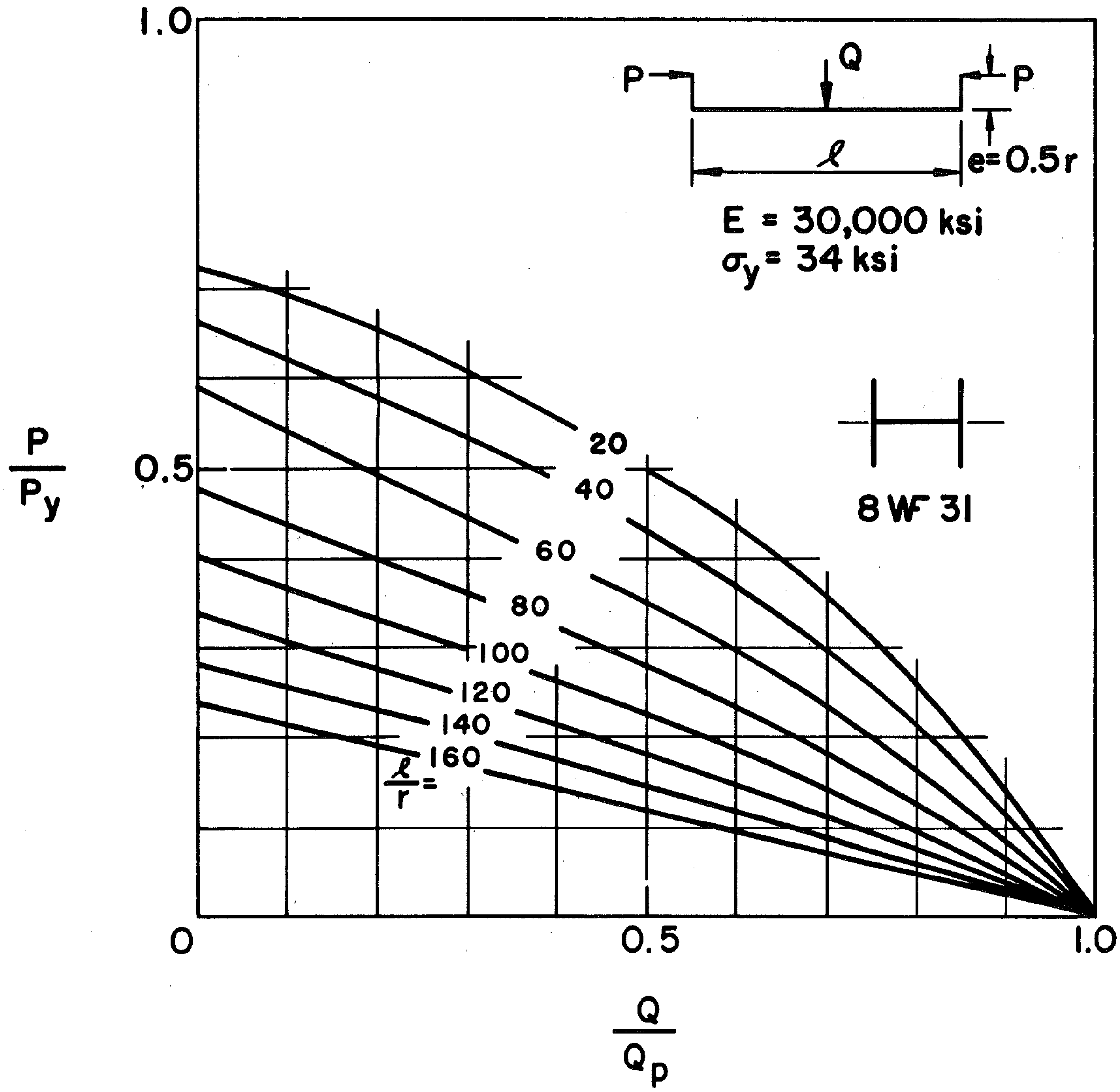


Fig. 10 Interaction Curves for a Wide-Flange Section Weak Axis Bending, Neglecting Influence of Residual Stress ($e/r = 0.5$)

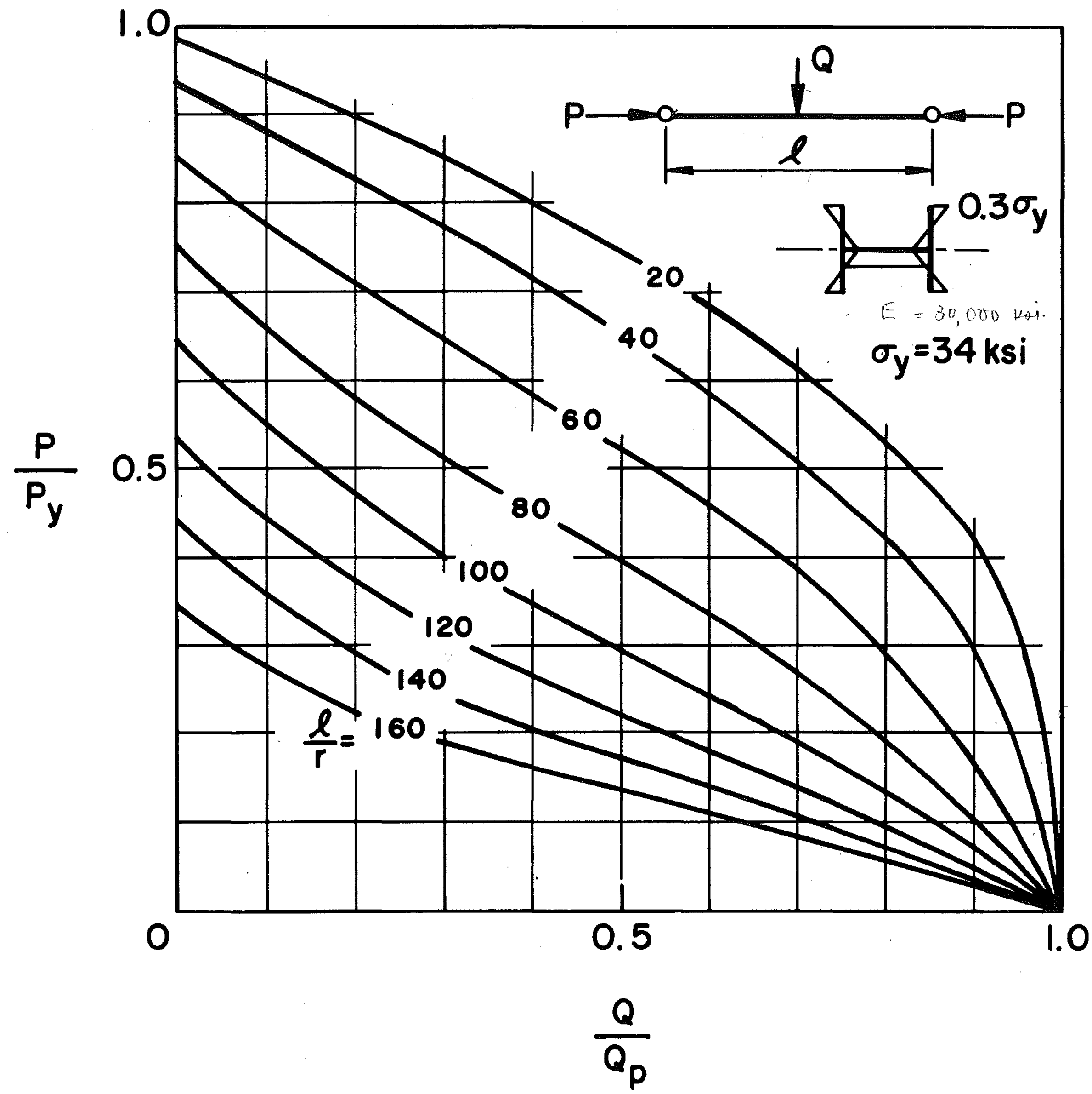


Fig. 11 Interaction Curves for a Wide-Flange Section, Weak Axis Bending, Including Influence of Residual Stress (e/r = 0)

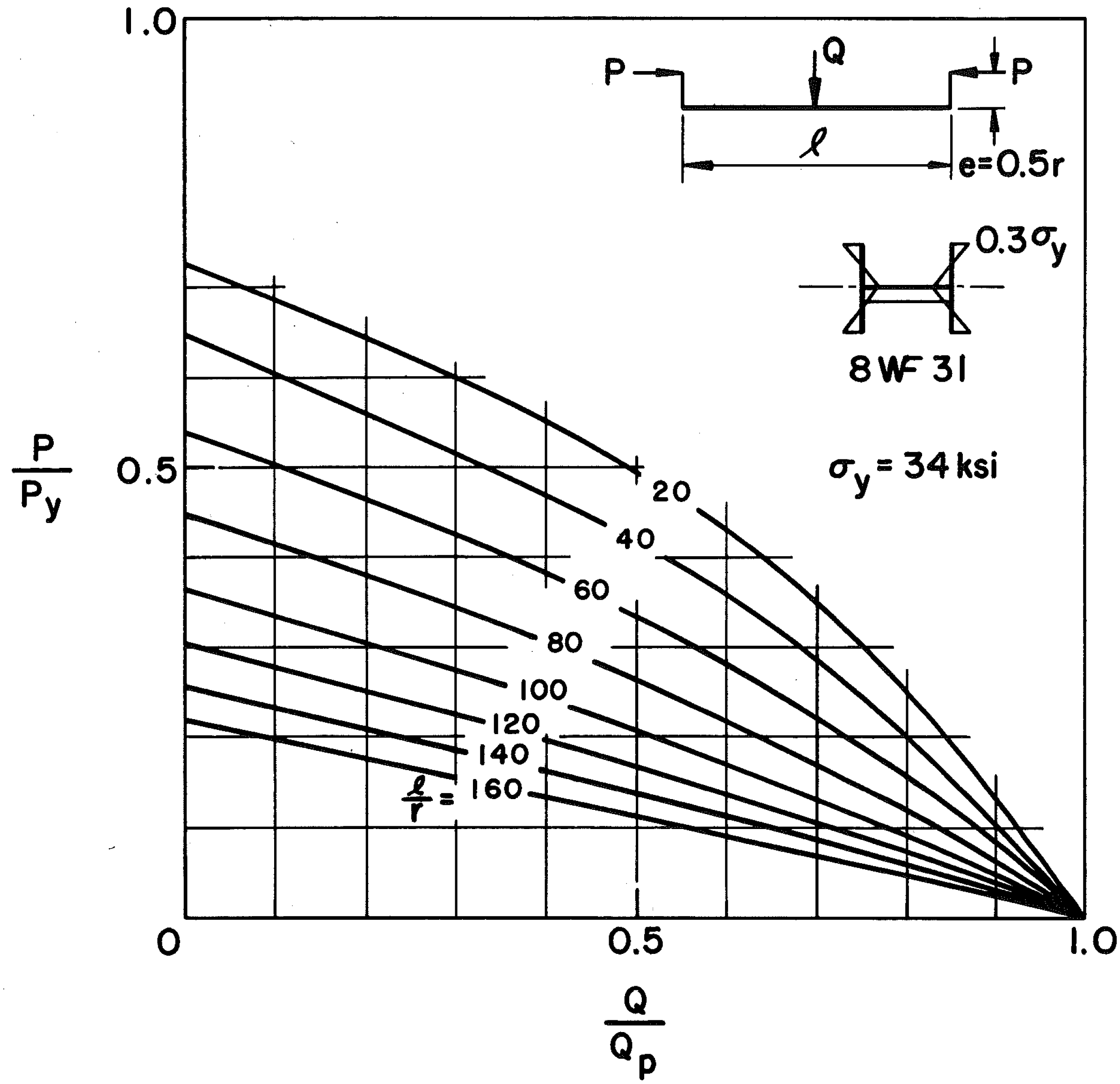


Fig. 12 Interaction Curves for a Wide-Flange Section,
Weak Axis Bending,
Including Influence of Residual Stress
(e/r = 0.5)

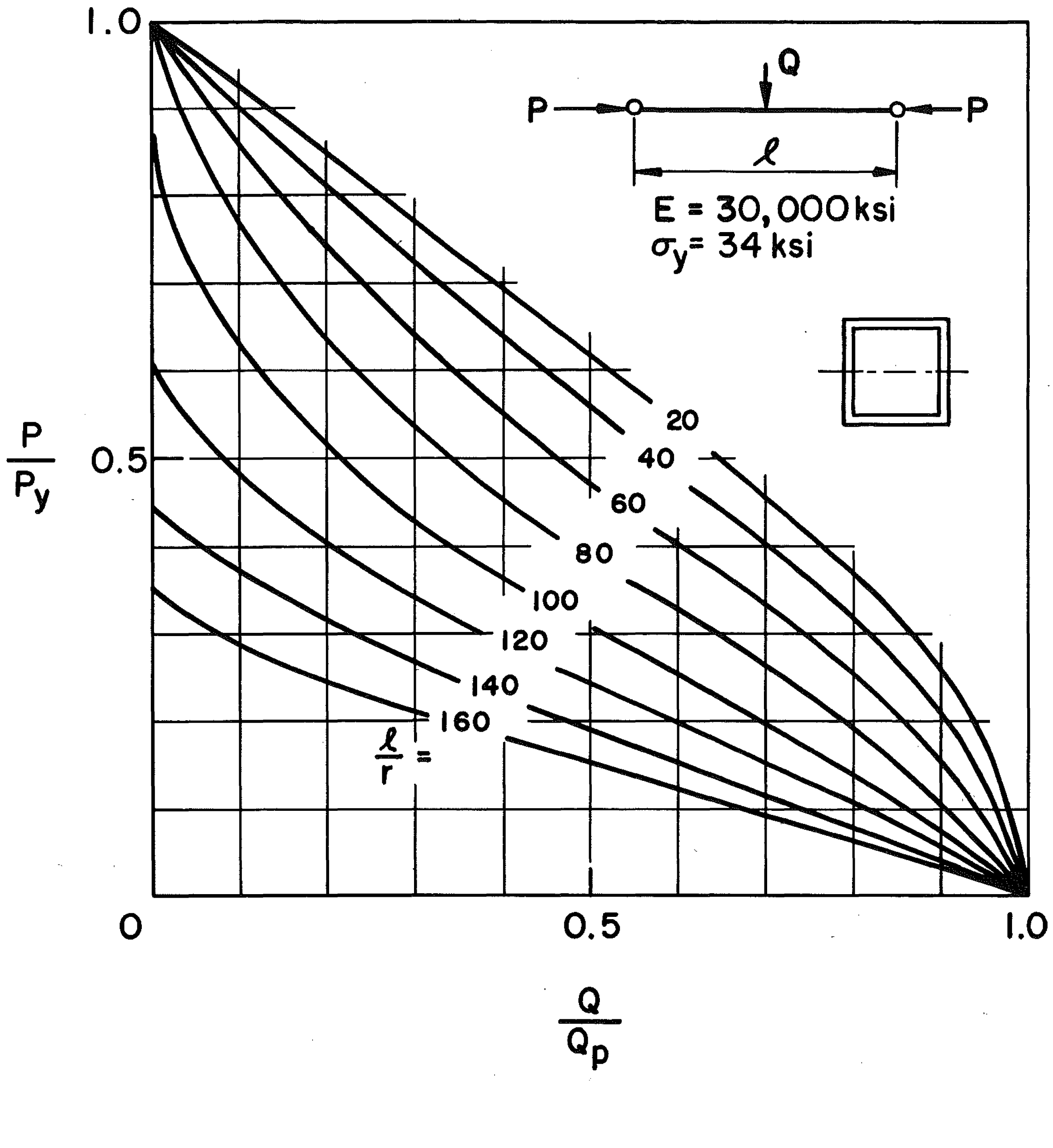


Fig. 13 Interaction Curves for a Square Tubular Section, Neglecting Influence of Residual Stress (e/r = 0)

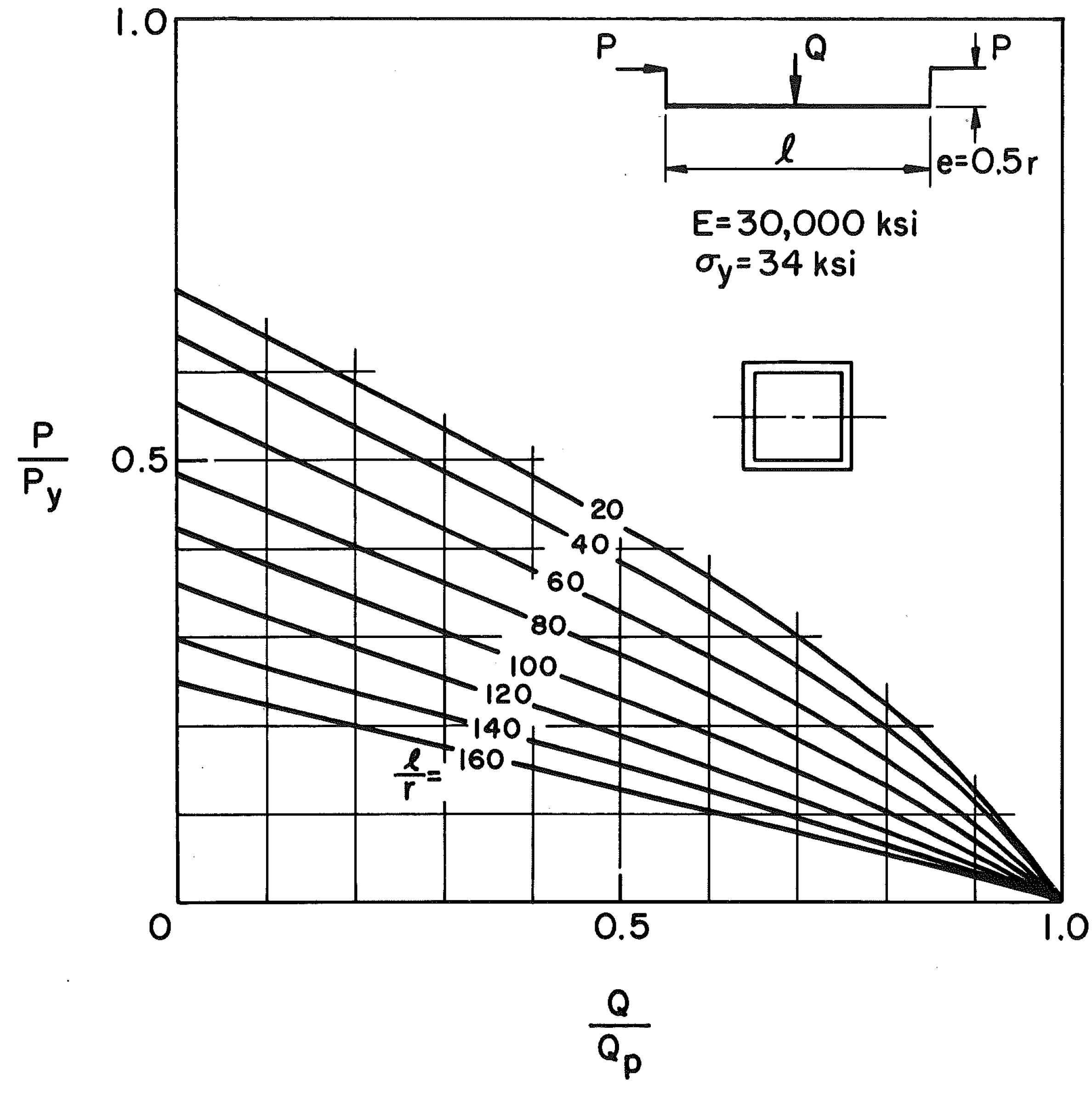


Fig. 14 Interaction Curves for a Square Tubular Section, Neglecting Influence of Residual Stress (e/r = 0.5)

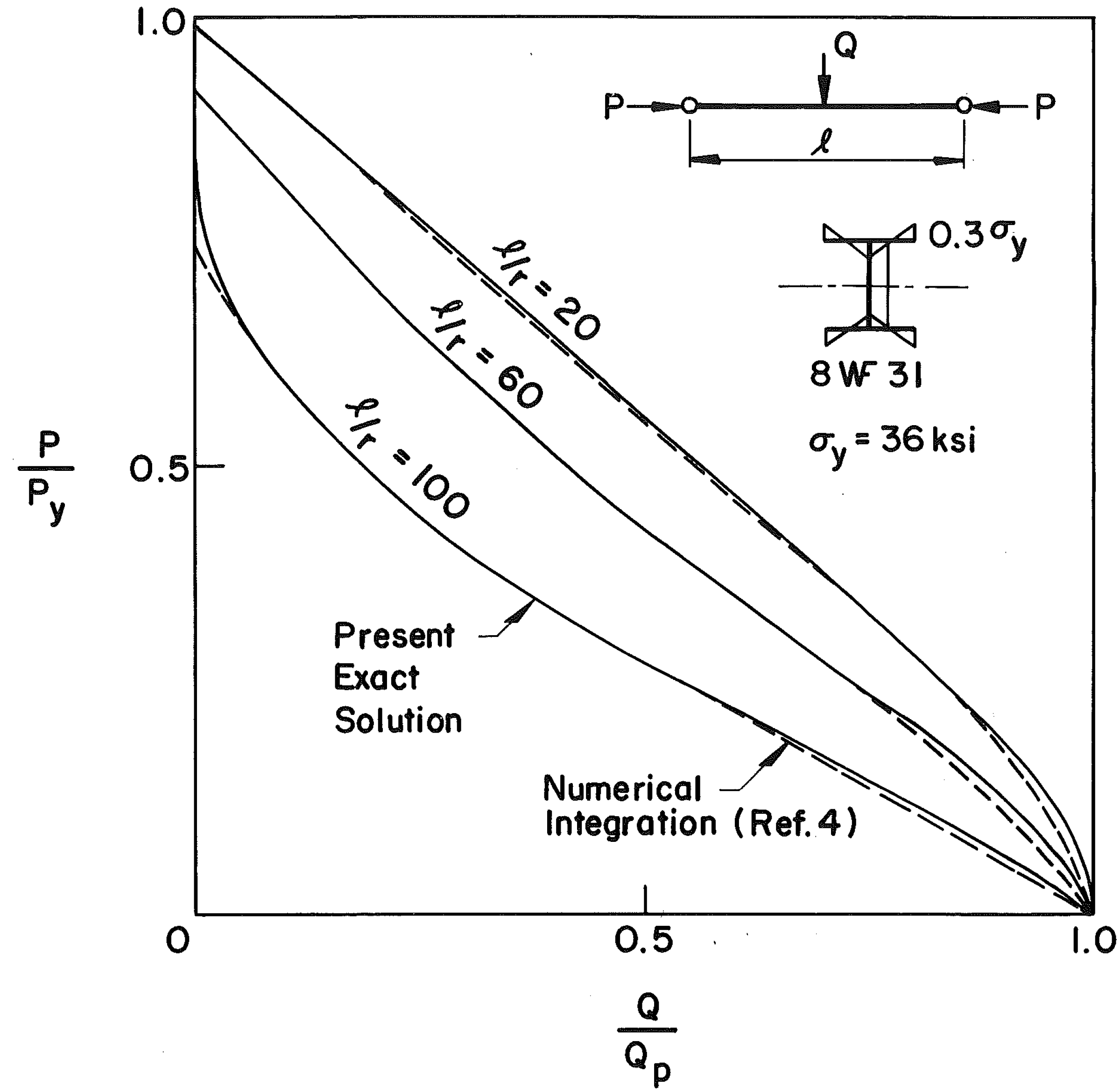


Fig. 15 Comparison Between "Present Analytical" and "Numerical Integration" Interaction Curves

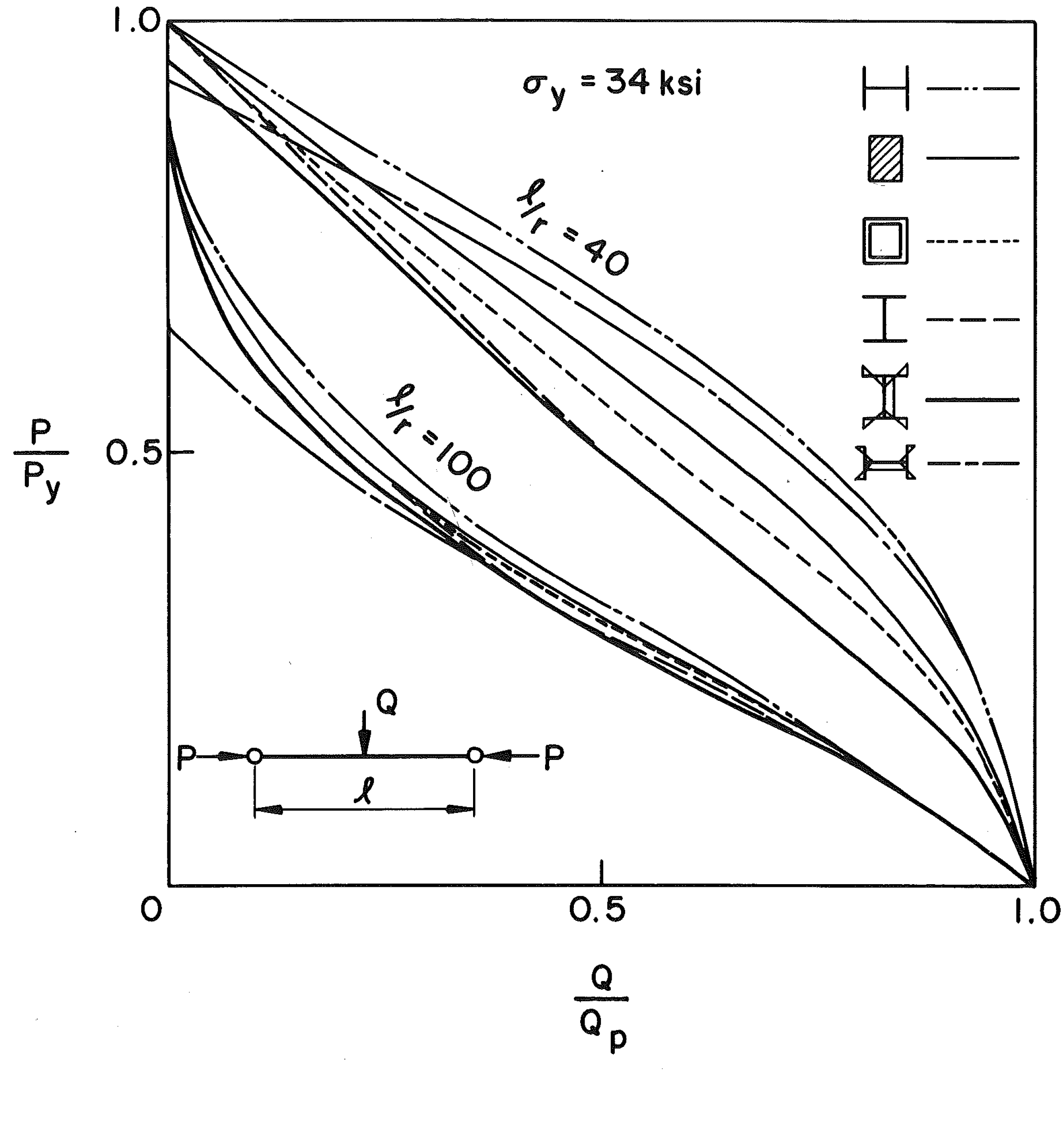


Fig. 16 Interaction Curves for Various Shapes of Beam-Column Section

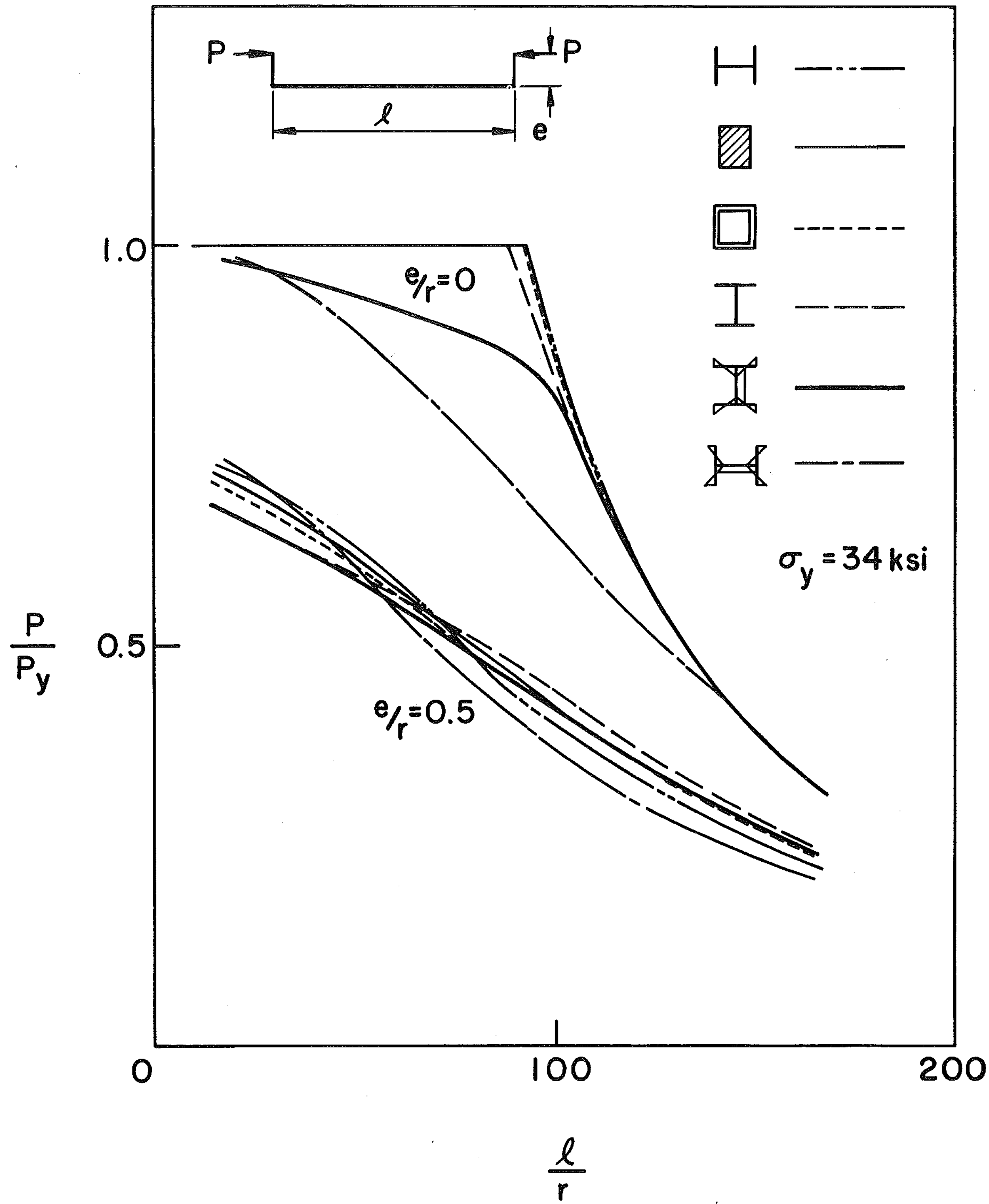


Fig. 17 Column Curves for Various Shapes of Column Cross Section

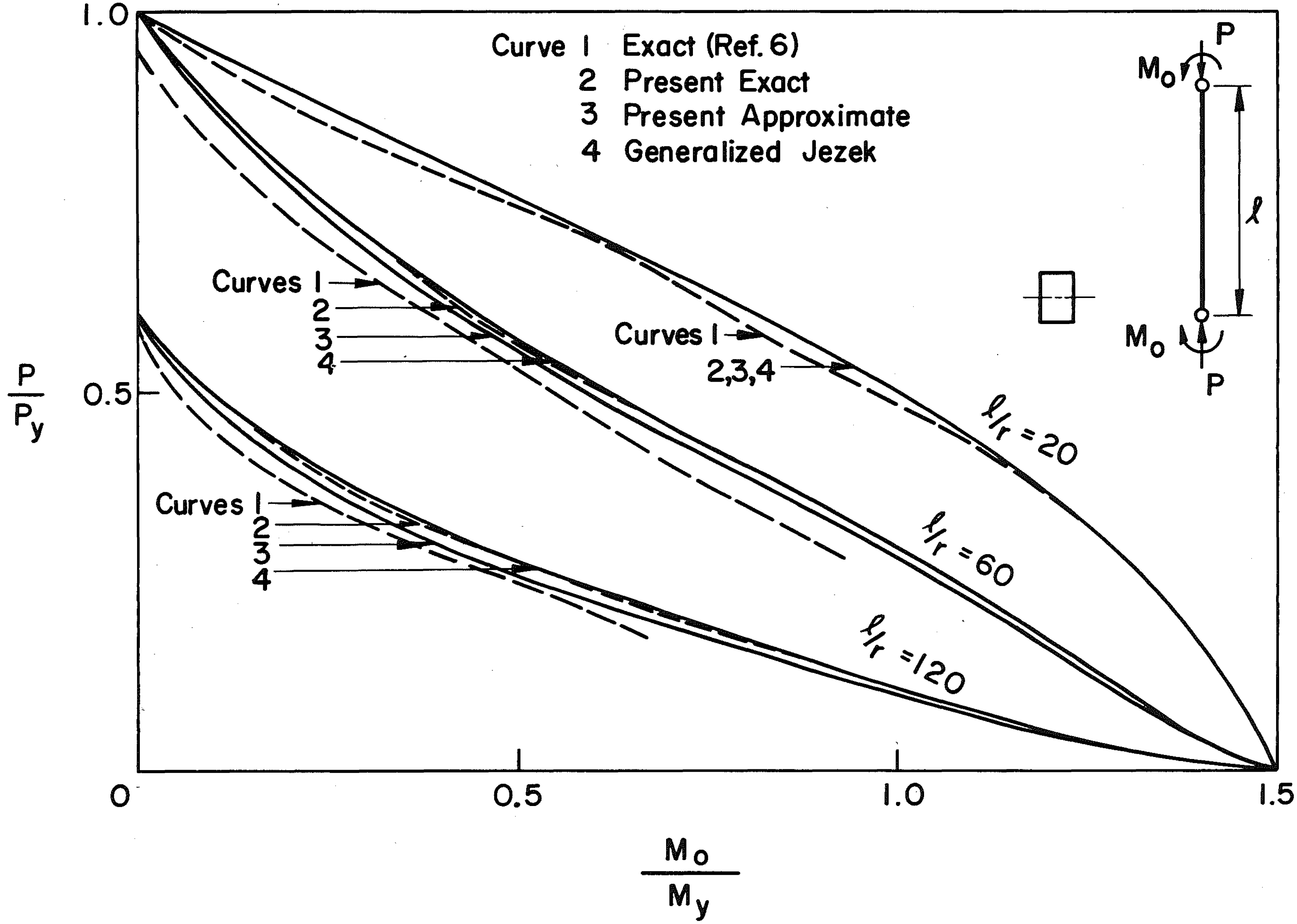


Fig. 18 Exact and Approximate Interaction Curves for a Solid Rectangular Section (Exact and Present Exact Curves Shown Dashed)

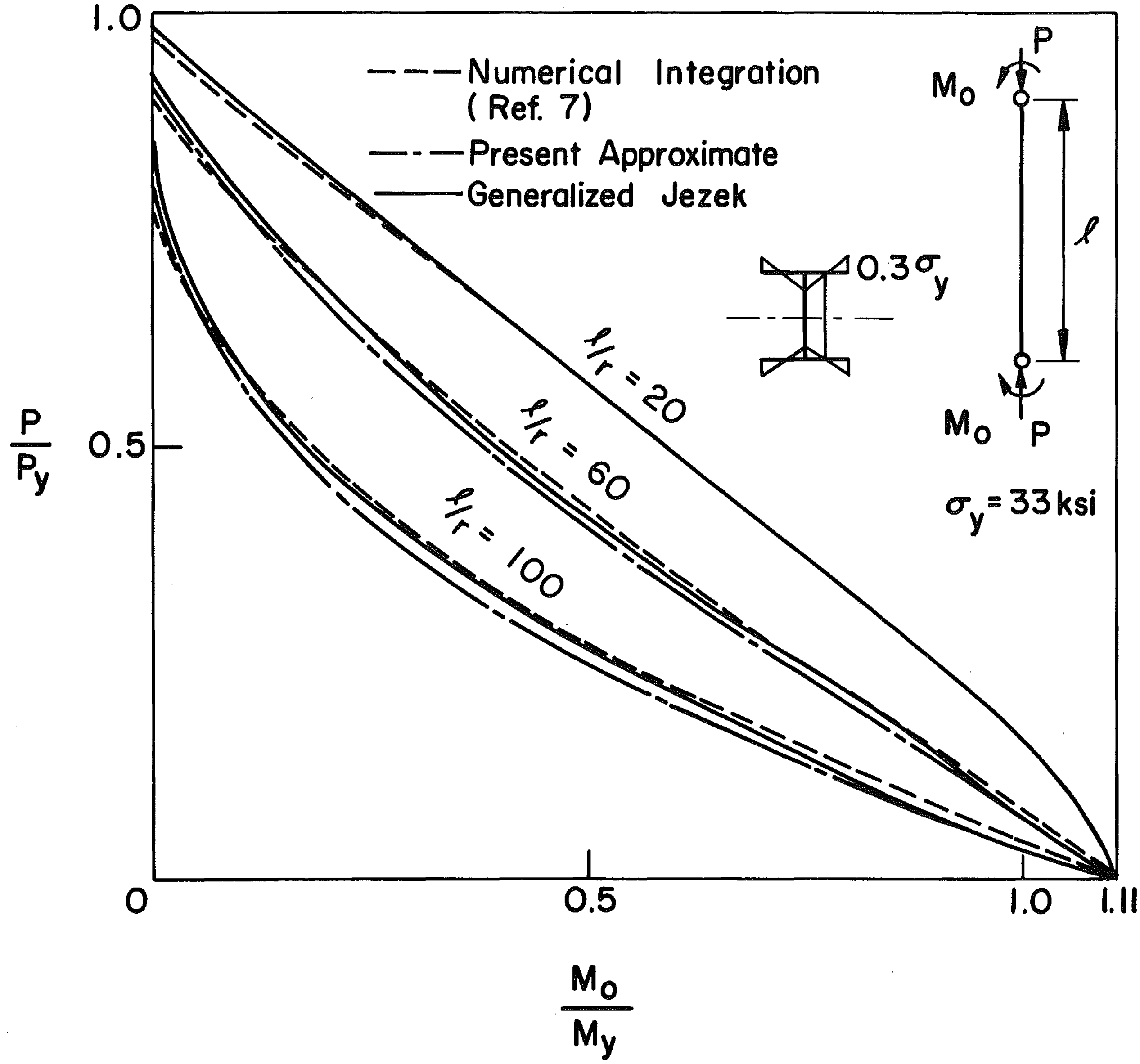


Fig. 19 Comparison Between "Exact" and "Approximate" Interaction Curves for a Wide-Flange Section, Including Influence of Residual Stress

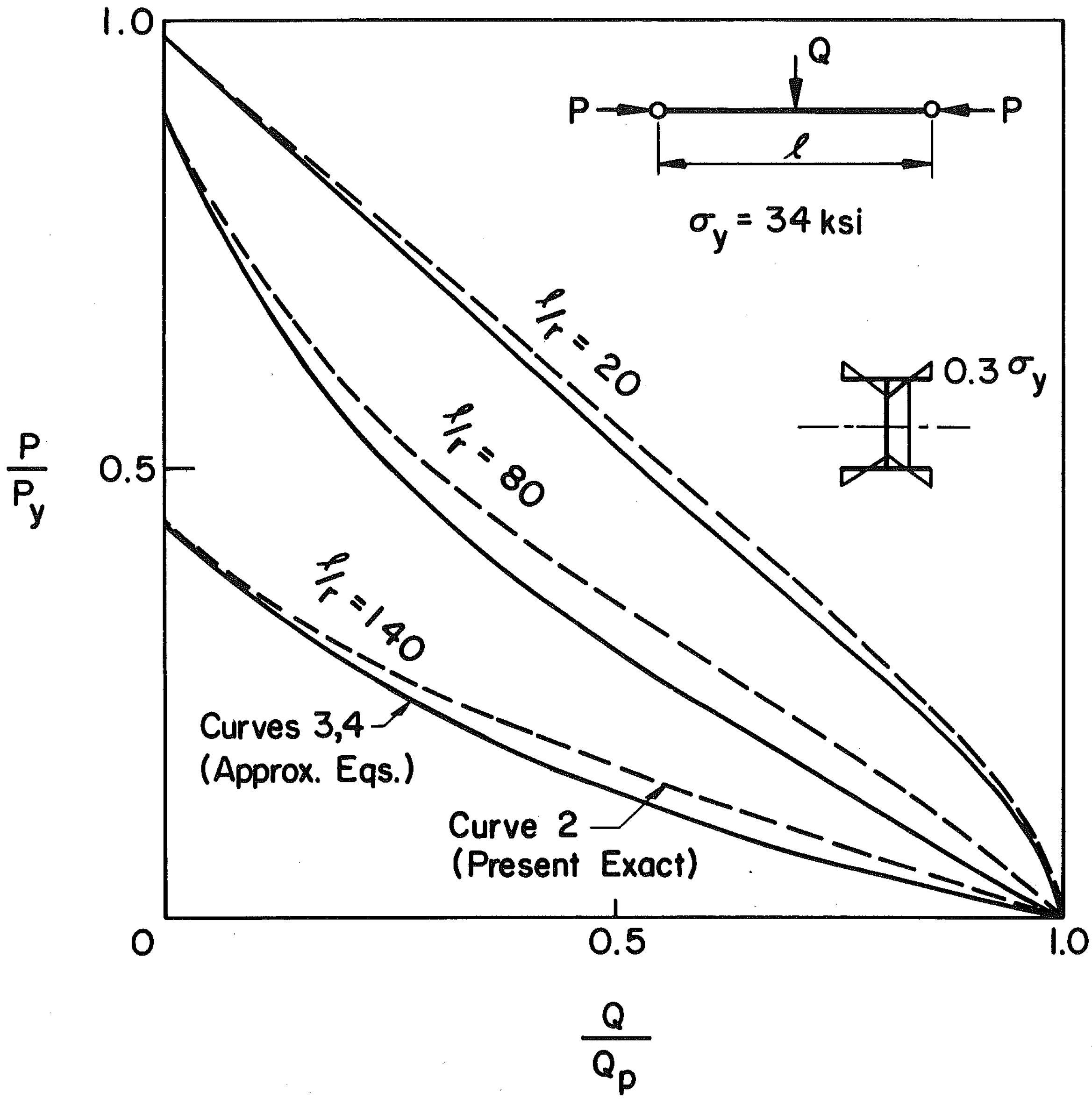


Fig. 20 Comparison Between Present "Exact" and "Approximate" Interaction Curves for a Wide-Flange Section with $e/r = 0$

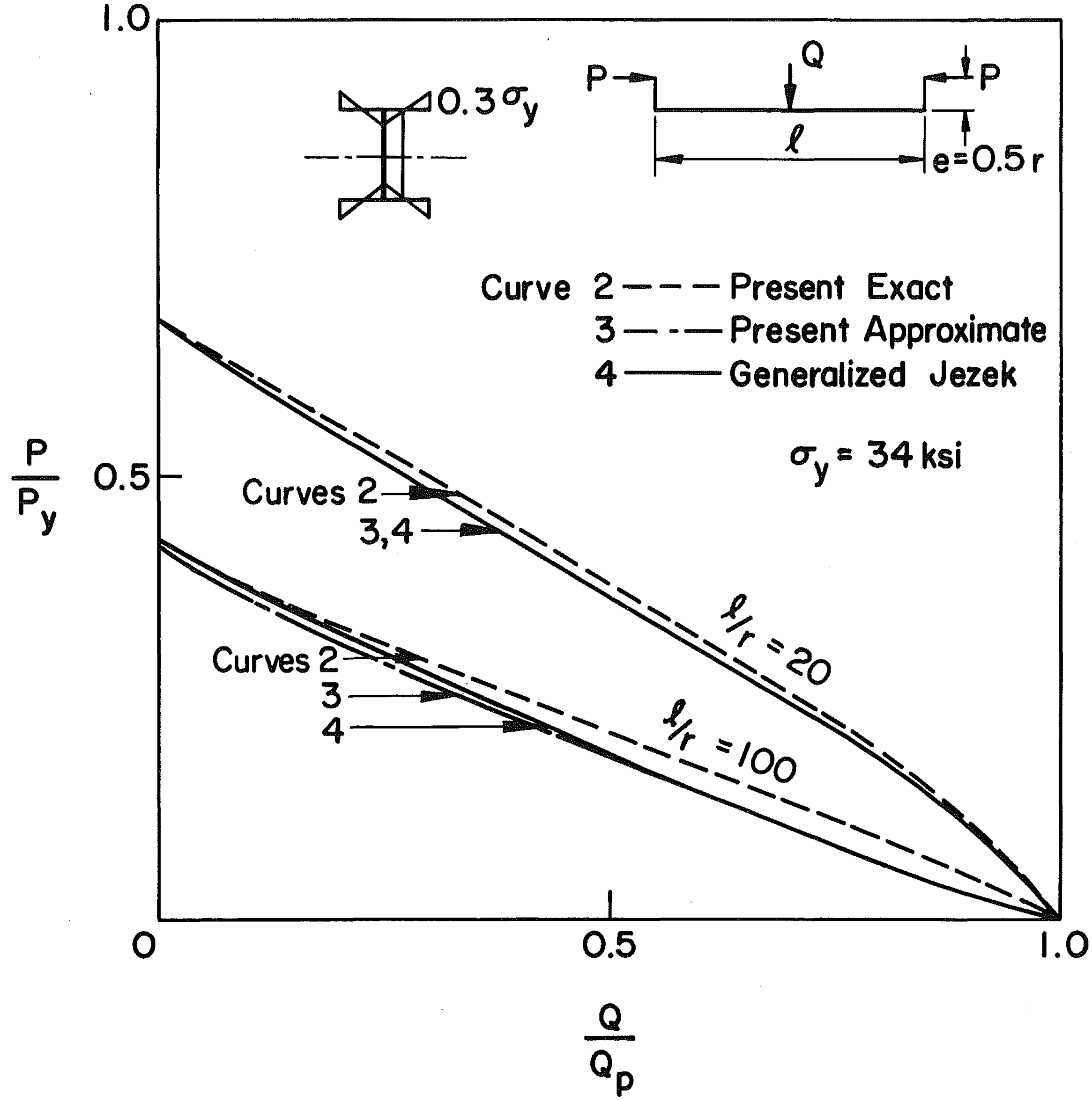


Fig. 21 Comparison Between Present "Exact" and "Approximate" Interaction Curves for a Wide-Flange Section with e/r = 0.5

10. APPENDIX

I. Additional M-Φ-P curves

Fig. 22 and Fig. 23

II. Additional M_o-P interaction curves

Fig. 24

III. Additional Q-P interaction curves

Fig. 25, Fig. 26, and Fig. 27

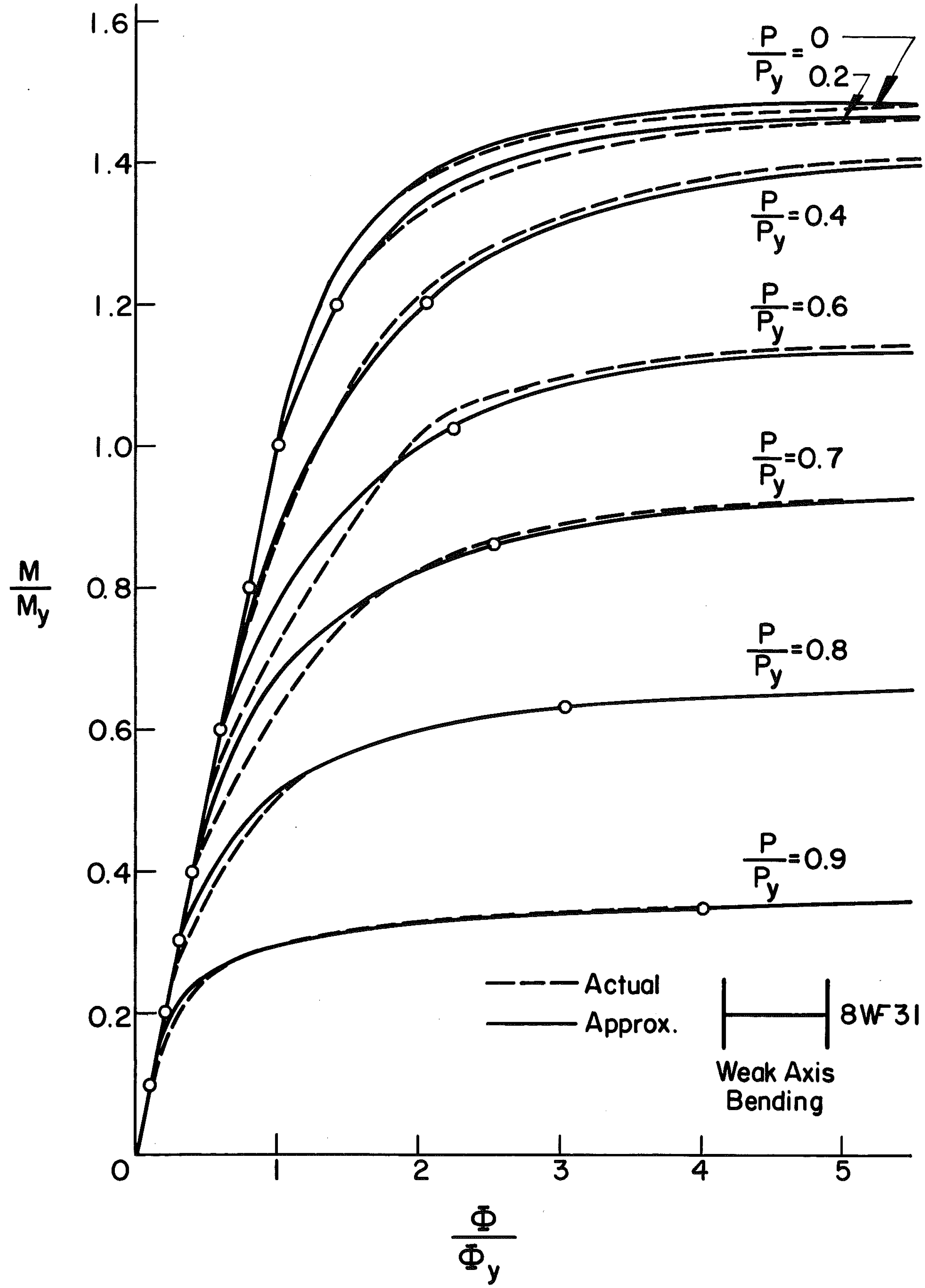


Fig. 22 Moment-Curvature-Thrust Relationship for a Wide-Flange Section (Actual Curves Shown Dashed)

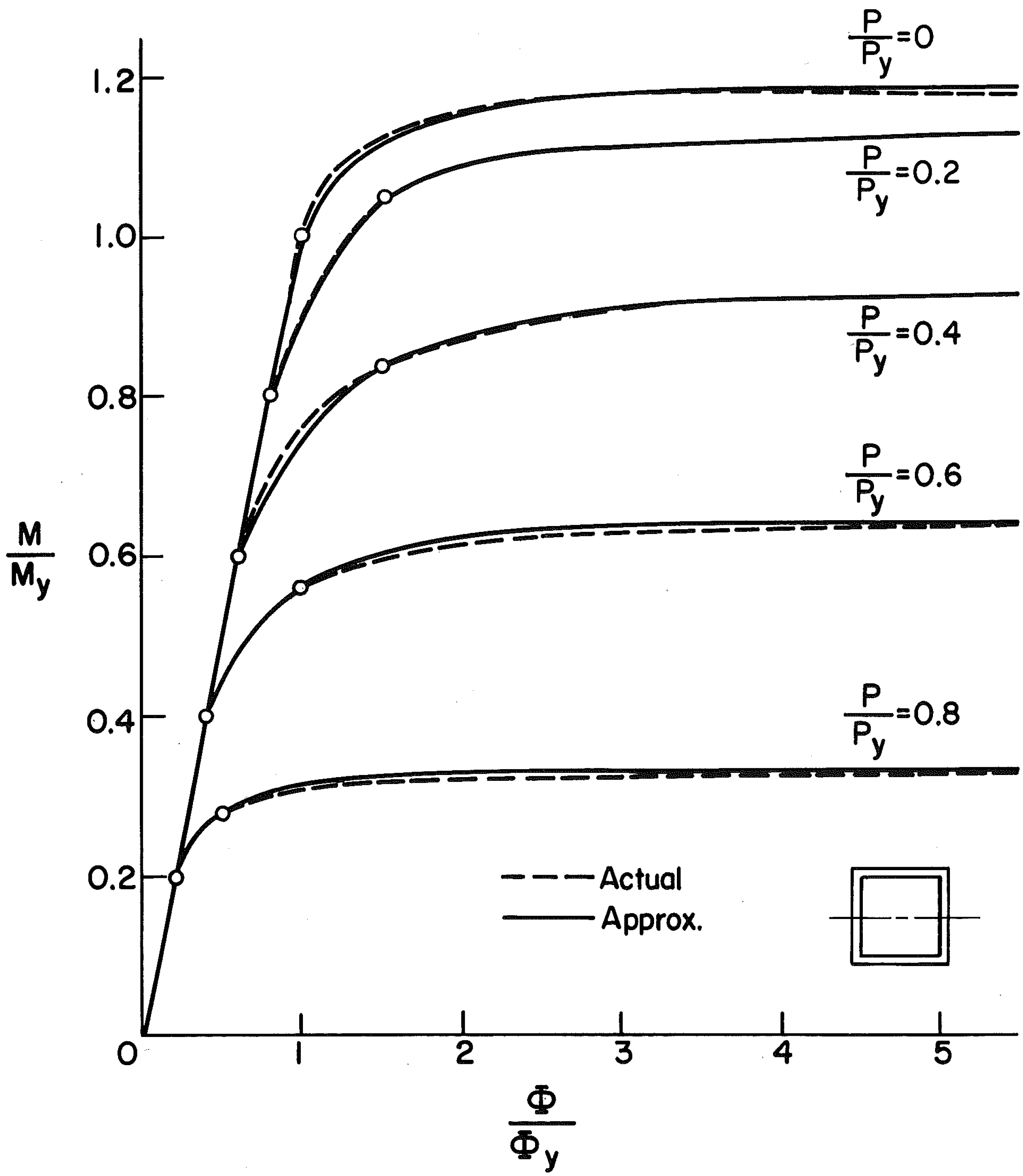


Fig. 23 Moment-Curvature-Thrust Relationship for a Square Tubular Section (Actual Curves Shown Dashed)

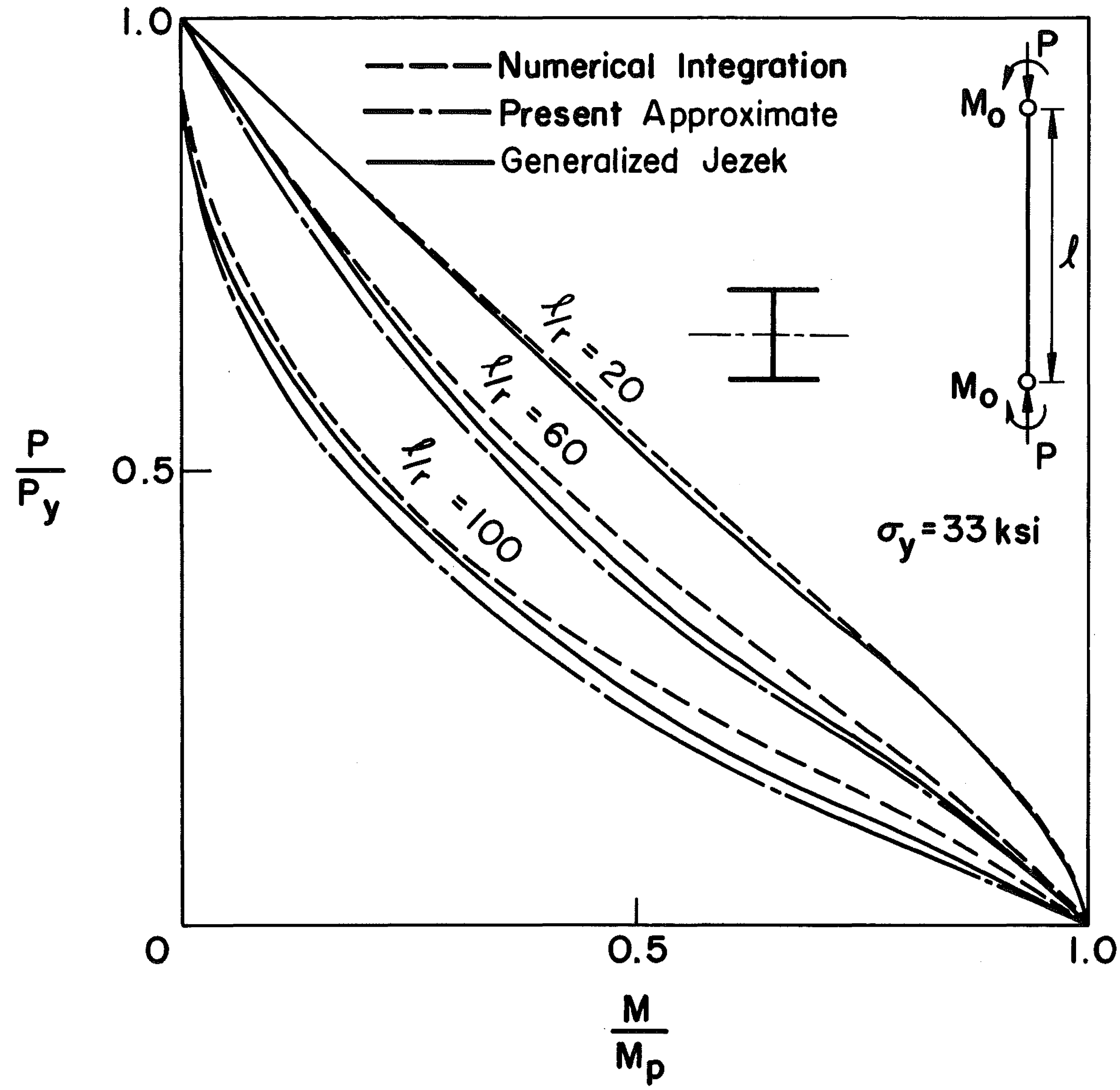


Fig. 24 Comparison Between "Exact" and "Approximate" Interaction Curves for a Wide-Flange Section, Neglecting Influence of Residual Stress

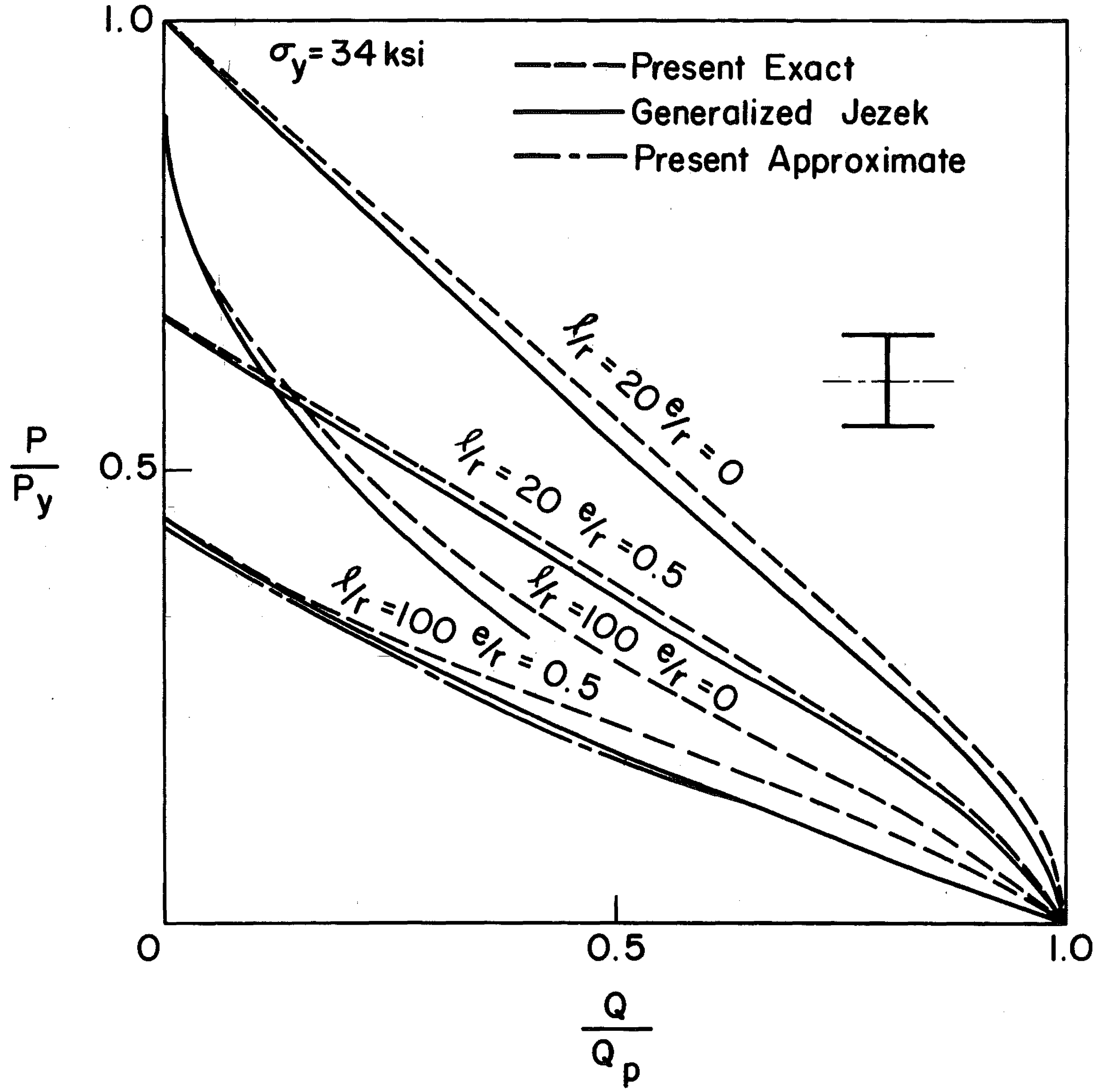


Fig. 25 Present "Exact" and "Approximate" Interaction Curves for a Wide-Flange Section with e/r = 0 and 0.5

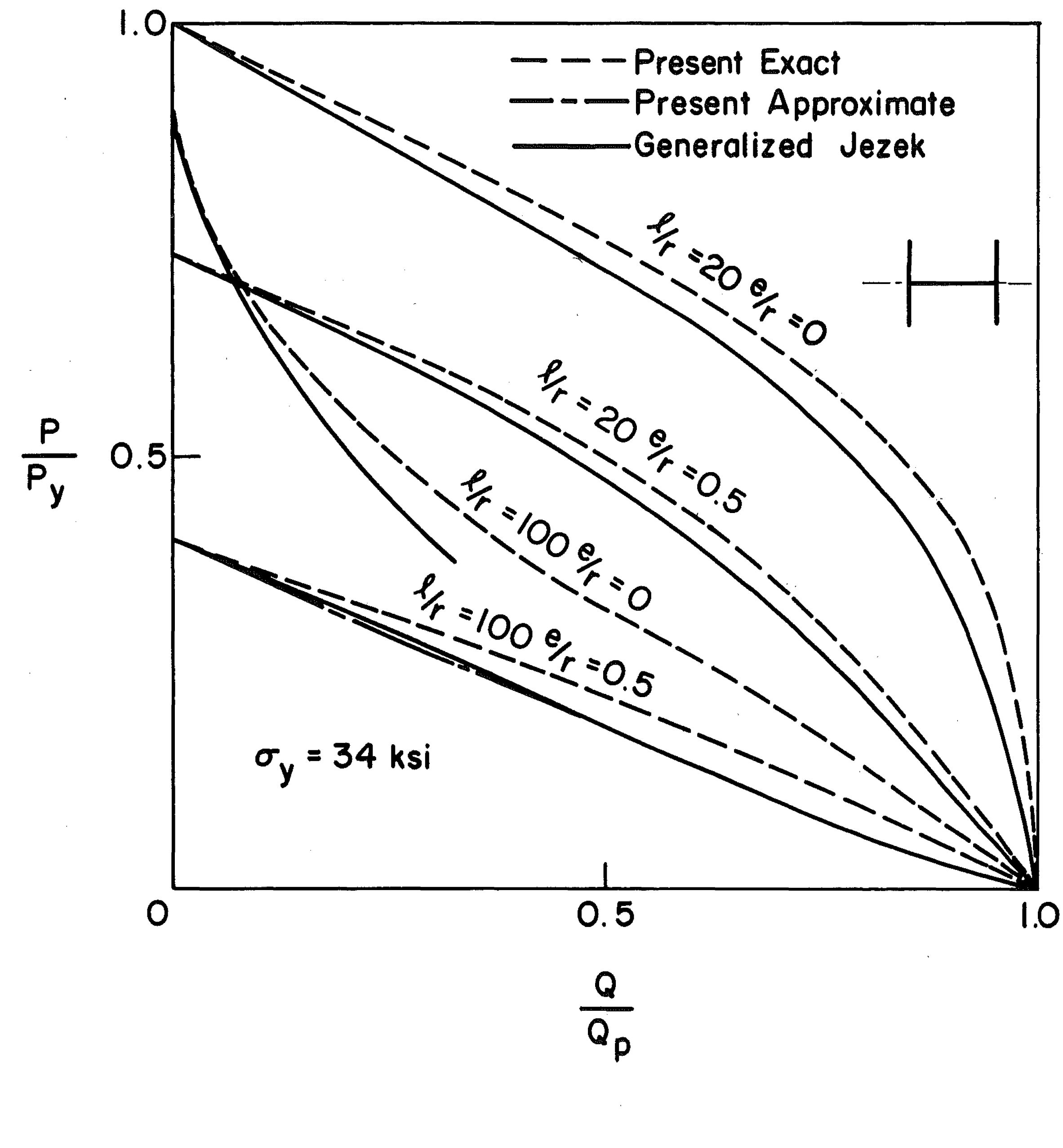


Fig. 26 Present "Exact" and "Approximate" Interaction Curves for a Wide-Flange Section, Weak Axis Bending, with e/r = 0 and 0.5

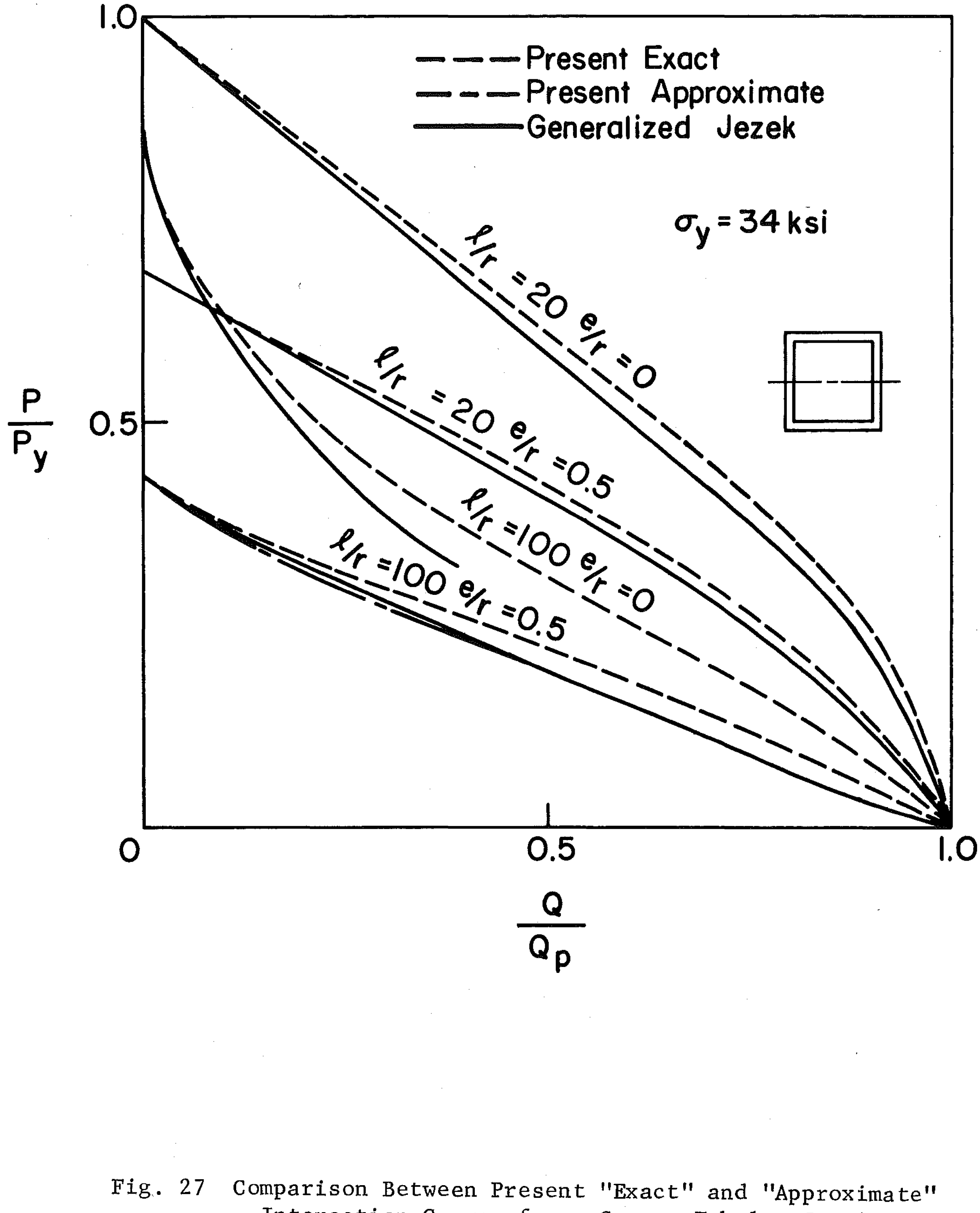


Fig. 27 Comparison Between Present "Exact" and "Approximate" Interaction Curves for a Square Tubular Section with e/r = 0 and 0.5

11. NOTATION

A	= area of section
$a, b, c, f, m_1, m_2, m_{pc}, \varphi_1, \varphi_2$	= arbitrary constants which define the generalized m-φ-p curve
D	= $[\ 1 + (\pi^2/8 - 1)\ p/p_E\ ]$, Eq. (48)
E	= modulus of elasticity , E = 30,000 ksi. (cf. p. 12.)
e	= eccentricity
h	= depth of section
I	= moment of inertia of section about the axis of bending
ℓ	= length of beam-column
M	= bending moment
M_m	= mid-height moment
M_o	= applied moment at the end of beam-column
M_y	= pure moment which causes first yielding in the section
m	= M/M_y
m_m	= M_m/M_y
m_o	= M_o/M_y
P	= axial thrust
P_y	= axial yield load
p	= P/P_y
p_E	= $\pi^2 E/\sigma_y\ (\ell/r)^2$, Eq. (33)

Q	= concentrated lateral load
Q_p	= plastic limit load according to simple plastic theory
q	= Q/Q_p
r	= radius of gyration about the axis of bending
S	= elastic section modulus
x,y	= coordinate axes, see Fig. 1
y_m	= mid-height deflection
α	= shape factor of section about the axis of bending
ϵ_y	= strain at yield point
σ_y	= yield stress
Φ	= curvature
Φ_o	= end curvature
Φ_m	= mid-height curvature
Φ_y	= curvature at initial yielding for pure bending moment
φ	= Φ/Φ_y
φ_m	= Φ_m/Φ_y

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13. ABSTRACT

Previous work on the solution of an eccentrically loaded beam-column under a concentrated load at the mid-span is extended. Expressions are first derived for the moment expressed explicitly in terms of curvature and thrust for various shapes of structural sections with or without the influence of residual stress. Using these results, interaction curves relating axial thrust, lateral load, and slenderness ratio with various values of end eccentricities are presented for the maximum load carrying capacity of the beam-columns. An approximate solution of the beam-column problem is then treated. Useful expressions for the interaction curves are obtained.

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14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
beams (supports); column (supports); curvature; deflection; elasticity; approximation; engineering mechanics; plasticity; stability						

Security Classification